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Applied string diagram rewriting

March 30, 2023

Algebraic rewriting seminar

Chapter 1:

Pro(p)s and diagrammatical theories

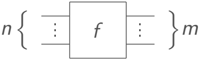
Pro(p)s

Pro(p)s

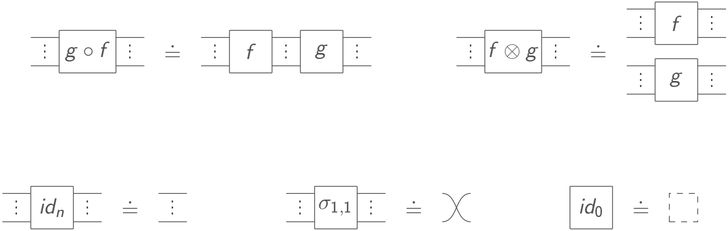
A (symmetric) strict monoidal category whose monoid of objects is $(\mathbb{N}, +)$.

- Tensor is denoted additively: $(0, +)$.
- A generating object 1.
- Objects of the form: $1 + \cdots + 1 \doteq n$.
- Arrows: $f : n \to m$, $h : m \to k$, $g : k \to \ell$.
- $id_n : n \to n$.
- $h \circ f : n \to k$.
- $f \otimes g : n + k \to m + \ell$.
- $\sigma_{n,m} : n + m \to m + n$.

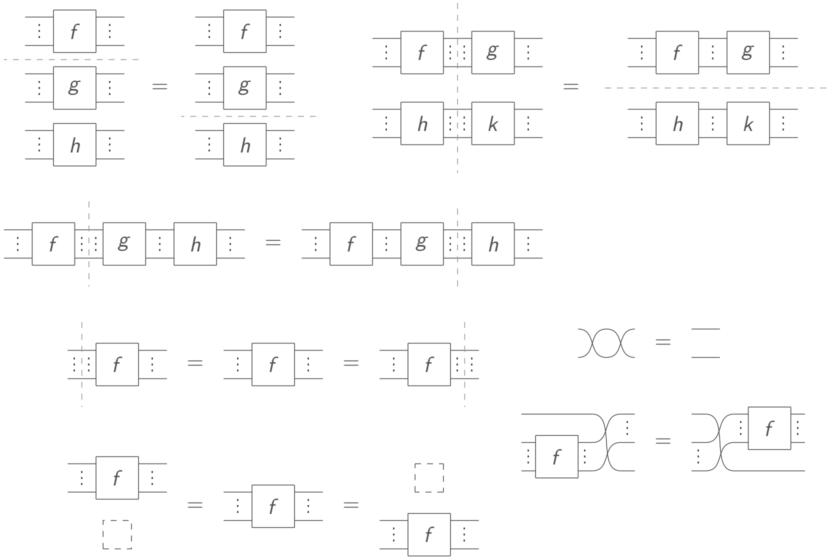
String diagrams



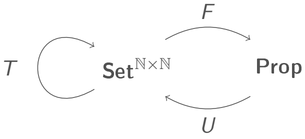
$f : n \to m$



Picturing tautologies



Equational theories



- $F : \mathbf{Set}^{\mathbb{N}\times\mathbb{N}} \to \mathbf{Prop}$, free diagrams over generators in Σ.
- $U : \mathbf{Prop} \to \mathbf{Set}^{\mathbb{N}\times\mathbb{N}}$ stores a collection of diagrams.
- **Signature:** $\Sigma \in \mathbf{Set}^{\mathbb{N}\times\mathbb{N}}$.
- **Equations:** $E \subseteq T(\Sigma) \times T(\Sigma)$.
- **Diagramatic language:** $F(\Sigma)/E$.

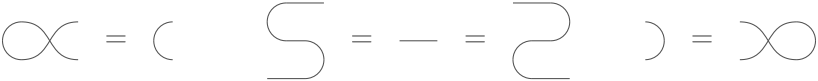
Interpretation

- Interpreting $F(\Sigma)/E$ in a prop $\mathbf{P}$.
- $[\![\_]\!] : F(\Sigma) \to \mathbf{P}$ prop morphism.
- **Soundness:** $[\![\_]\!] : F(\Sigma)/E \to \mathbf{P}$.
- **Universality:** $[\![\_]\!]$ is full.
- **Completeness:** $[\![\_]\!]$ is faithful.

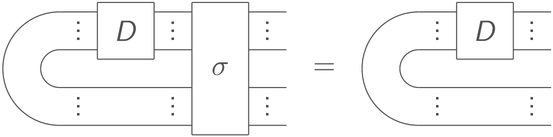
Flexsymmetry

$\eta : 0 \to 2$

$\epsilon : 2 \to 0$



A diagram $D : n \to m$ is said to be **flexsymmetric** if for all permutations $\sigma \in \mathfrak{S}_{n+m}$:



The permutation σ is obtained by composing swaps.

Chapter 2:

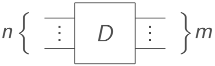
Graphical Linear Algebra

Linear relations

A linear relation $\mathcal{R} : n \to m$ is a linear subspaces of $\mathbb{R}^n \oplus \mathbb{R}^m$.

- $0 \to 0$: only $\{0\}$.
- $0 \to 1$: $\{0\}$ and $\mathbb{R}$.
- $1 \to 1$: it's $\mathbb{R} \cup \{\infty, \perp, \top\}$.
- A linear map $A : n \to m$ embedds as:
  $\{x \oplus Ax, x \in \mathbb{R}^n\} \subseteq \mathbb{R}^n \oplus \mathbb{R}^m$.
- Composition as usual relations.
- Tensor with $\oplus$.

Spiders and wires



$\xrightarrow{[\![\cdot]\!]}$ $\mathcal{R} \subseteq \mathbb{R}^n \oplus \mathbb{R}^m$

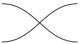
$D : n \to m$

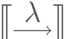
$[\![$ $]\!] = \{x \oplus x, x \in \mathbb{R}\}$

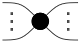
$[\![$  $]\!] = \{0 \oplus (x, x), x \in \mathbb{R}\}$

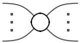
$[\![$  $]\!] = \{0\}$

$[\![$  $]\!] = \{(x, x) \oplus 0, x \in \mathbb{R}\}$

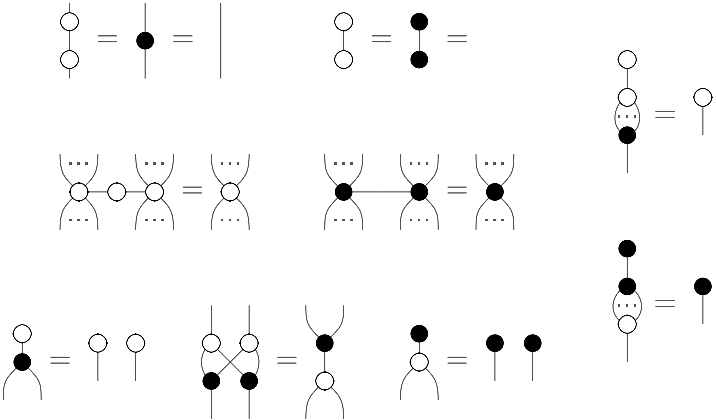
$[\![$  $]\!] = \{(x, y) \oplus (y, x), x, y \in \mathbb{R}\}$

$[\![$  $]\!] = \{x \oplus \lambda x, x \in \mathbb{R}\}$

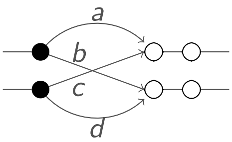
$[\![$  $]\!] = \{(x, \ldots, x) \oplus (x, \ldots, x), x \in \mathbb{R}\}$

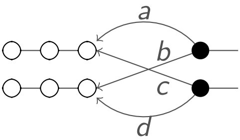
$[\![$  $]\!] = \{\overrightarrow{x} \oplus \overrightarrow{y}, \sum_i x_i + \sum_j y_i = 0, \overrightarrow{x} \in \mathbb{R}^n, \overrightarrow{y} \in \mathbb{R}^m\}$

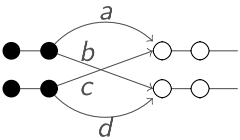
Rules

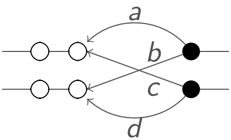


Matrices

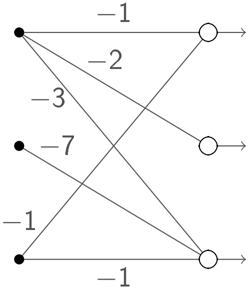
- The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$: 

- Its kernel: 

- Its range: 

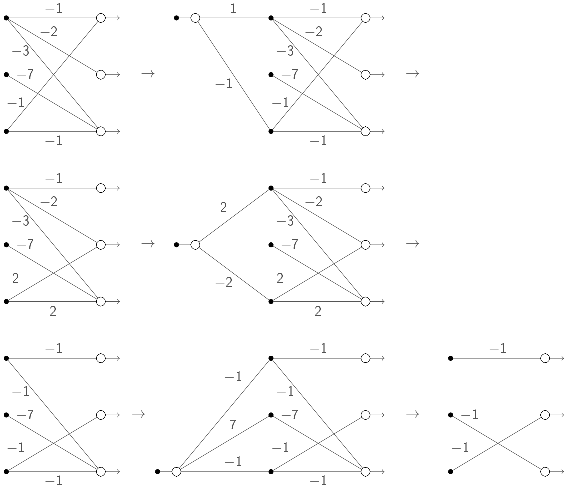
- Its inverse: 

Span normal form



It corresponds to the subspace spanned by: $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 7 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}.$

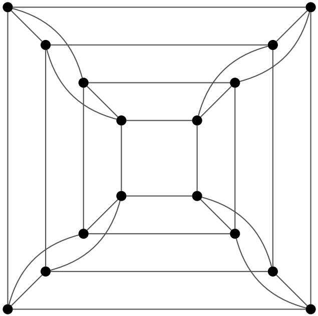
Gauss pivot



Chapter 3:

Perfect matchings

A graph

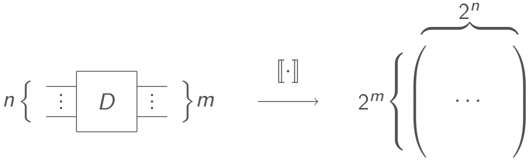


Linear maps

An arrow $L : n \to m$ is a linear map $\mathbb{R}^{2^n} \to \mathbb{R}^{2^n}$.

- $0 \to 0$: it's $\mathcal{R}$.
- $0 \to 1$: it's $\mathbb{R}^2$.
- $1 \to 1$: it's $\mathcal{M}_{2\times 2}(\mathbb{R})$.
- Composition by matrix product.
- Tensor with $\otimes$.

Spiders and wires

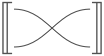


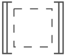
$D : n \to m$

$[\![D]\!] \in \mathcal{M}_{2^m \times 2^n}(\mathbb{C})$

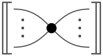

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

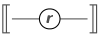

$= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$


$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$


$= (1)$

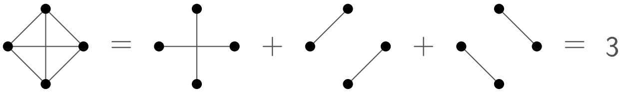

$= \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}$


$= \sum_{|x \cdot y| = 1} |x\rangle\langle y|$



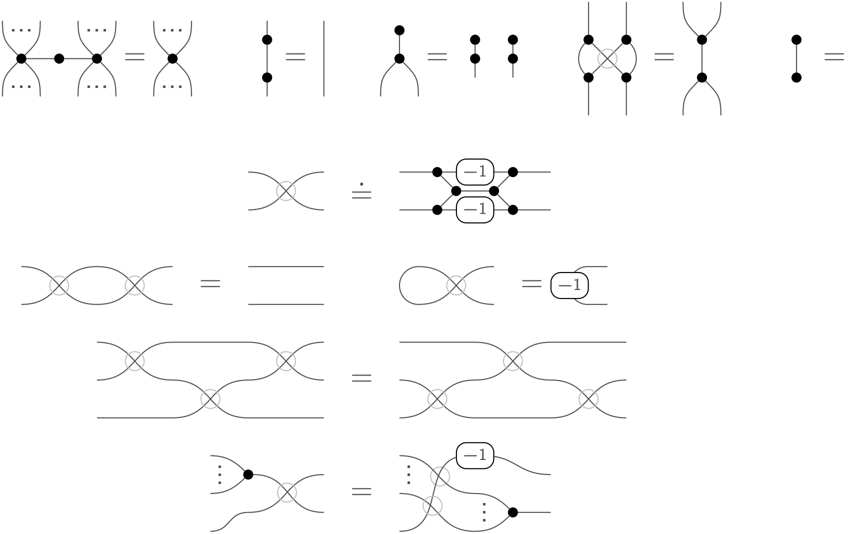
$= \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}$

An invariant

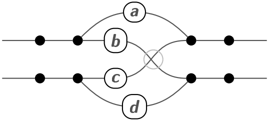


- This counts the perfect matchings.
- In general, weighted sum over all the perfect matchings.

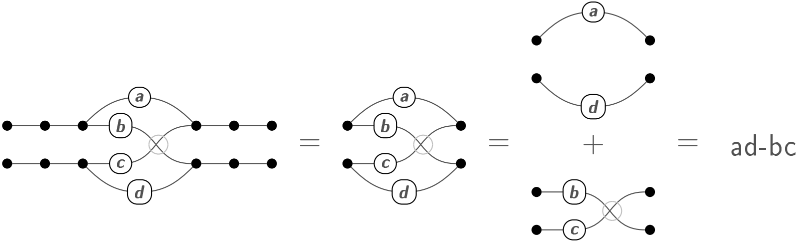
The fermionic swap and some rules



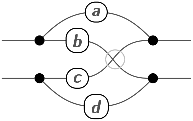
Matrices are back!



Determinant



- $\sum_{\sigma} \varepsilon(\sigma) \prod_{i} A_{i,\sigma(i)} = \det(A)$.



- Inverse and cofactor formula.

Conclusion

- If a compact closed prop admits a flexsymmetric axiomatisation, then graph rewriting technics apply.
- Many flexsymmetric diagrammatical languages exist. I just gave a few examples.
- Historical applications are quantum computing.
- Many things still have to be done!