Diagrammatic rewriting and categorification

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Séminaire de réécriture algébrique

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Outline

I. Linear 2-categories and string diagrams

II. Diagrammatic rewriting and linear $(3,2)$-polygraphs

III. Catégorification of $U_q(\mathfrak{sl}_2)$

Categorification and decategorification

- **Categorification** is a concept introduced by Crane and Frenkel in low dimensional topology. It consists in replacing set theoretic notions by categorical ones:

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relation between elements	morphism of objects
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  - **Example:** Consider the set $\mathbb{N}$ of natural numbers. A 'categorification' of $\mathbb{N}$ is given by the category **FinSet** of finite sets via cardinality.
  - Sum and product in $\mathbb{N}$ correspond to disjoint union and cartesian product in **FinSet** respectively.
  - $+$ and $\times$ in $\mathbb{N}$ satisfy commutativity, associativity and distributivity, but $\sqcup$ and $\times$ in **FinSet** satisfy such laws only up to natural isomorphisms.

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- The reverse process, called **decategorification** is made via the Grothendieck group.
- If $\mathcal{A}$ is an additive category (*i.e.* a 1-category equipped with finite biproducts $\oplus : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$), the **Grothendieck group** $K_0(\mathcal{A})$ of $\mathcal{A}$ is the free abelian group with basis the isomorphism classes $[M]$ of 0-cells of $\mathcal{A}$ quotiented by the subgroup generated by the emements

$$[A_1] - [A_2] + [A_3] \text{ for every } 0\text{-cells } A_1, A_2, A_3 \text{ of } \mathcal{A} \text{ such that } A_2 \cong A_1 \oplus A_3.$$

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- **Example:** Category $\mathbf{Vect}_{\mathbb{K}}$ of $\mathbb{K}$-vector spaces. Then $K_0(\mathbf{Vect}_{\mathbb{K}}) \cong \mathbb{Z}$. Indeed, consider the map

$$f : \mathbf{Vect}_{\mathbb{K}} \to \mathbb{Z}, \; V \mapsto \dim(V)$$

Categorification and decategorification

- **Gradings:** Let R be a $\mathbb{Z}$-graded ring. Consider the category $R - \mathbf{gMod}$ of graded R-modules. Denote by $\{1\}$ the shift of grading:
$$\text{For } M = \bigoplus_{i \in \mathbb{Z}} M_i, \quad (M\{1\})_j = M_{j+1}.$$
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- If $A = \bigoplus_{i,j \in I} A_{i,j}$ admits a 1-categorical structure, one will categorify A using an additive 2-category, that is a 2-category $\mathcal{A}$ such that for every 0-cells x and y, $\mathcal{A}(x, y)$ is an additive category.

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- The Grothendieck group of an additive 2-category $\mathcal{A}$ is the 1-category $K_0(\mathcal{A})$ whose:
  - 0-cells are the 0-cells of $\mathcal{A}$,
  - 1-cells with source A and target B are the elements of $K_0(\mathcal{A}_1(A, B))$. Composition of 1-cells is defined by
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- Given the algebra $A = \bigoplus_{i,j\in I} A_{i,j}$, one will construct a 2-category $\mathcal{A}$ with 0-cells the elements of I and such that the 1-categories $\mathcal{A}(i,j)$ are in correspondence with the $A_{i,j}$. Then, prove that
$$A \cong K_0(\mathcal{A}).$$
- Proving such an isomorphism is a difficult task in general. A relevant question to do so is to **compute bases for the spaces of morphisms of $\mathcal{A}$**.

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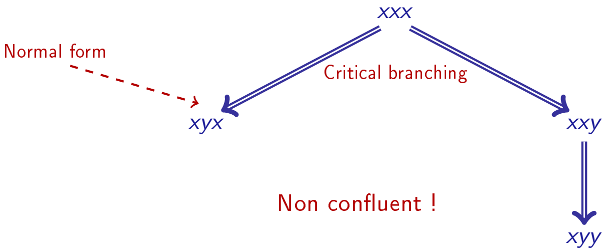

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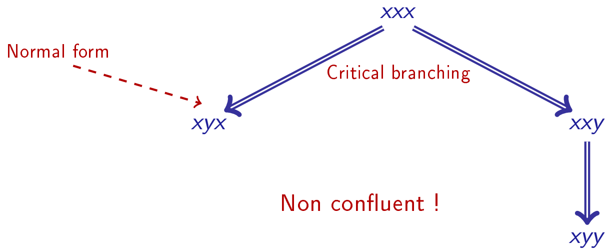

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$$\begin{tikzcd}[column sep=small] & yzx - [z,y]x \arrow[r] & yxz - y[z,x] - [z,y]x \arrow[r] & xyz - [y,x]z - y[z,x] - [z,y]x \\ zyx \arrow[ur] \arrow[dr] & & & \\ & zxy - z[y,x] \arrow[r] & xzy - [z,x]y - z[y,x] \arrow[r] & xyz - x[z,y], [z,x]y - z[y,x] \end{tikzcd}$$

Linear bases from convergence

- **Example:** Associative algebra A presented by generators $X = \{x,\ y\}$ and relations $R = \{x^2 \Rightarrow xy\}$.

Normal form

Critical branching

$$\begin{tikzcd} & xxx \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\ xyx & & xxy \arrow[d, Rightarrow] \\ & & xyy \end{tikzcd}$$

Non confluent !

  - It is **terminating**, using a deglex order on $x > y$.
  - The monomials in normal form **span** the algebra. They are not linearly independant: the missing property is **confluence**.

- **Poincaré-Birkhoff-Witt theorem:** Let $\mathcal{L}$ be a Lie algebra and let X be a totally well-ordered basis of $\mathcal{L}$. Then, the universal enveloping algebra $U(\mathcal{L})$ of $\mathcal{L}$ admits as a basis

$$\left\{ x_1^{\alpha_1} \ldots x_k^{\alpha_k} \mid \quad x_i < x_{i+1} \in X,\ \alpha_i \in \mathbb{N} \right\}$$

  - presentation of $U(\mathcal{L})$: $\{X \mid yx - xy - [y,x], \quad x \neq y \in X\}$
  - choice of orientation of relations: $yx \to xy + [y,x]$, where $x < y$
  - this rewriting system is terminating, using a degree lexicographic order on $x_1 < x_2 < \cdots < x_k$,
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I. Linear 2-categories and string diagrams

Monoidal categories

- A **monoidal category** is a 1-category $(\mathcal{C}_0, \mathcal{C}_1)$ equipped with
  - a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called tensor product,
  - a unit object $\mathbf{1} \in \mathcal{C}_0$, called unit object,
  - a natural isomorphism a satisfying

$$\begin{tikzcd}
\mathcal{C} \times \mathcal{C} \times \mathcal{C} \arrow[r, "\otimes \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \otimes"'] & \mathcal{C} \times \mathcal{C} \arrow[d, "\otimes"] \arrow[dl, phantom, "\Swarrow a"] \\
\mathcal{C} \otimes \mathcal{C} \arrow[r, "\otimes"'] & \mathcal{C}
\end{tikzcd}$$

  - natural isormorphisms λ and ρ satisfying

$$\begin{tikzcd}
1 \times \mathcal{C} \arrow[r, "\mathbf{1} \times \mathrm{id}"] \arrow[dr, "\sim"'] & \mathcal{C} \times \mathcal{C} \arrow[d, "\otimes"] \arrow[dl, phantom, "\Swarrow \lambda"] \arrow[dr, phantom, "\Searrow \rho"] & \mathcal{C} \times 1 \arrow[l, "\mathrm{id} \otimes \mathbf{1}"'] \arrow[dl, "\sim"] \\
& \mathcal{C} &
\end{tikzcd}$$

where 1 is the 1-category with one object $\bullet$ and one morphism $\mathrm{id}_\bullet$, that are sent via $\mathbf{1}$ onto $\mathbf{1}$ and $\mathrm{id}_\mathbf{1}$.

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- For every objects $x, y, z \in \mathcal{C}_0$, there are isomorphisms

$$a_{x,y,z} : (x \otimes y) \otimes z \to x \otimes (y \otimes z), \quad \lambda_x : \mathbf{1} \otimes x \to x, \quad \rho_x : x \otimes \mathbf{1} \to x.$$

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& \mathcal{C} &
\end{tikzcd}$$

$$\lambda, \quad \rho$$

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- A monoidal category is **strict** when the natural isomorphisms a, λ and ρ are identities.

- Composition of morphism and tensor products in a monoidal category satisfy **exchange law**, that is for every $f, g, h, k \in \mathcal{C}_1$,

$$(f \otimes g) \circ (h \otimes k) = (f \circ g) \otimes (g \circ k).$$

Linear categories

- Let $\mathcal{V} = (V, \otimes, \mathbf{1}, a, \lambda, \rho)$ be a monoidal category. A category **enriched over** $\mathcal{V}$ is a category $\mathcal{C} = (\mathcal{C}_0, \mathcal{C}_1)$ such that:
  - for every $x, y \in \mathcal{C}_0$, $\mathcal{C}(x, y) := \mathrm{Hom}_{\mathcal{C}}(x, y)$ is an object of $\mathcal{V}$,
  - for every $x, y, z \in \mathcal{C}_0$; $\circ_{x,y,z} : \mathcal{C}(x, y) \otimes \mathcal{C}(y, z)$ is a morphism of $\mathcal{V}$,
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  such that the following diagrams commute in $\mathcal{V}$:

$$
\begin{tikzcd}
 & & (\mathcal{C}(x,y) \otimes \mathcal{C}(y,z)) \otimes \mathcal{C}(z,t) \arrow[dll, "\circ_{x,y,z} \otimes \mathbf{id}"'] \arrow[drr, "a"] & & \\
\mathcal{C}(x,z) \otimes \mathcal{C}(z,t) \arrow[dr, "\circ_{x,z,t}"'] & & & & \mathcal{C}(x,z) \otimes (\mathcal{C}(y,z) \otimes \mathcal{C}(z,t)) \arrow[dl, "\mathbf{id} \otimes \circ_{y,z,t}"] \\
 & \mathcal{C}(x,t) & & \mathcal{C}(x,y) \otimes \mathcal{C}(y,z) \arrow[ll, "\circ_{x,y,t}"] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{C}(x,y) \otimes \mathcal{C}(y,y) \arrow[r, "\circ_{x,y,y}"] & \mathcal{C}(x,y) & \mathcal{C}(x,x) \otimes \mathcal{C}(x,y) \arrow[l, "\circ_{x,x,y}"'] \\
\mathcal{C}(x,y) \otimes \mathbf{1} \arrow[u, "\mathbf{id} \otimes \iota_y"] \arrow[ur, "\rho"'] & & \mathbf{1} \otimes \mathcal{C}(x,y) \arrow[u, "\iota_x \otimes \mathbf{id}"'] \arrow[ul, "\lambda"]
\end{tikzcd}
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 & (\mathcal{C}(x,y) \otimes \mathcal{C}(y,z)) \otimes \mathcal{C}(z,t) \arrow[dl, "\circ_{x,y,z} \otimes \mathrm{id}"'] \arrow[dr, "a"] & \\
\mathcal{C}(x,z) \otimes \mathcal{C}(z,t) \arrow[d, "\circ_{x,z,t}"'] & & \mathcal{C}(x,z) \otimes (\mathcal{C}(y,z) \otimes \mathcal{C}(z,t)) \arrow[d, "\mathrm{id} \otimes \circ_{y,z,t}"] \\
\mathcal{C}(x,t) & & \mathcal{C}(x,y) \otimes \mathcal{C}(y,z) \arrow[ll, "\circ_{x,y,t}"]
\end{tikzcd}$$

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\mathcal{C}(x,y) \otimes \mathcal{C}(y,y) \arrow[r, "\circ_{x,y,y}"] & \mathcal{C}(x,y) & \mathcal{C}(x,x) \otimes \mathcal{C}(x,y) \arrow[l, "\circ_{x,x,y}"'] \\
\mathcal{C}(x,y) \otimes \mathbf{1} \arrow[u, "\mathrm{id} \otimes \iota_y"] \arrow[ur, "\rho"'] & & \mathbf{1} \otimes \mathcal{C}(x,y) \arrow[u, "\iota_x \otimes \mathrm{id}"'] \arrow[ul, "\lambda"]
\end{tikzcd}$$

- A $\mathbb{K}$-**linear category** is a category enriched over $(\mathbf{Vect}_{\mathbb{K}}, \otimes, \mathbb{K}, a, \lambda, \rho)$, that is for every $x, y \in \mathcal{C}_0$, $\mathcal{C}(x, y)$ is a $\mathbb{K}$-vector space, and composition of morphisms $\mathcal{C}(x, y) \times \mathcal{C}(y, z) \xrightarrow{\circ} \mathcal{C}(x, z)$ is bilinear:

$$f \circ (\lambda g + \mu h) = \lambda(f \circ g) + \mu(f \circ h).$$

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\begin{tikzcd}
 & (\mathcal{C}(x,y) \otimes \mathcal{C}(y,z)) \otimes \mathcal{C}(z,t) \arrow[dl, "\circ_{x,y,z}\otimes \mathbf{id}"'] \arrow[drr, "a"] & & \\
\mathcal{C}(x,z) \otimes \mathcal{C}(z,t) \arrow[dr, "\circ_{x,z,t}"'] & & & \mathcal{C}(x,z) \otimes (\mathcal{C}(y,z) \otimes \mathcal{C}(z,t)) \arrow[dl, "\mathbf{id}\otimes\circ_{y,z,t}"] \\
 & \mathcal{C}(x,t) & \mathcal{C}(x,y) \otimes \mathcal{C}(y,z) \arrow[l, "\circ_{x,y,t}"] &
\end{tikzcd}
$$

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$$f \circ (\lambda g + \mu h) = \lambda (f \circ g) + \mu (f \circ h).$$

- A $\mathbb{K}$-**linear monoidal category** is a monoidal category in which the tensor product of morphisms $\otimes : \mathcal{C}(x,y) \times \mathcal{C}(z,t) \to \mathcal{C}(x \otimes z, y \otimes t)$ is $\mathbb{K}$-bilinear:

$$f \otimes (\lambda g + \mu h) = \lambda (f \otimes g) + \mu (f \otimes h).$$

Linear 2-categories

- Recall that a 2-category is a category enriched over $(\mathbf{Cat}_1, \times, \begin{tikzcd} \bullet \arrow[loop above, "1_\bullet"] \end{tikzcd})$. Explicitely, we have a set $\mathcal{C}_0$ of objects, and for $p, q \in \mathcal{C}_0$, $\mathcal{C}(p,q)$ is a 1-category.
  - objects of $\mathcal{C}(p,q)$ are 1-cells with source p and target q. We denote by $\mathcal{C}_1$ the set of all 1-cells.
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- There are two compositions in a 2-category:

$$\begin{tikzcd} x \arrow[r, bend left=50, "p"{name=U}] \arrow[r, bend right=50, "q"'{name=D}] \arrow[from=U, to=D, Rightarrow, "f"] & y \arrow[r, bend left=50, "p'"{name=U2}] \arrow[r, bend right=50, "q'"'{name=D2}] \arrow[from=U2, to=D2, Rightarrow, "g"] & z \end{tikzcd} \rightsquigarrow \begin{tikzcd} x \arrow[r, bend left=50, "q \star_0 q'"{name=U}] \arrow[r, bend right=50, "p' \star_0 p'"'{name=D}] \arrow[from=U, to=D, Rightarrow, "f \star_0 g"] & z \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, bend left=70, "p"{name=U}] \arrow[r, "q"{name=M}] \arrow[r, bend right=70, "r"'{name=D}] \arrow[from=U, to=M, Rightarrow, "f"] \arrow[from=M, to=D, Rightarrow, "g"] & y \end{tikzcd} \rightsquigarrow \begin{tikzcd} x \arrow[r, bend left=50, "p"{name=U}] \arrow[r, bend right=50, "r"'{name=D}] \arrow[from=U, to=D, Rightarrow, "f \star_1 g"] & y \end{tikzcd}$$

- These compositions satisfy the **exchange law**: for every f, f', g, g' in $\mathcal{C}_2$:

$$(f \star_1 f') \star_0 (g \star_1 g') = (f \star_0 g) \star_1 (f' \star_0 g').$$

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$$
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x \arrow[r, bend left, "p"] \arrow[r, bend right, "q"'] \arrow[r, phantom, "\Downarrow f"] & y \arrow[r, bend left, "p'"] \arrow[r, bend right, "q'"'] \arrow[r, phantom, "\Downarrow g"] & z
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
x \arrow[r, bend left, "q\star_0 q'"] \arrow[r, bend right, "p'\star_0 p'"'] \arrow[r, phantom, "\Downarrow f\star_0 g"] & z
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[r, bend left=60, "p"] \arrow[r, "q" description] \arrow[r, bend right=60, "r"'] \arrow[r, phantom, shift left=3, "\Downarrow f"] \arrow[r, phantom, shift right=3, "\Downarrow g"] & y
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
x \arrow[r, bend left, "p"] \arrow[r, bend right, "r"'] \arrow[r, phantom, "\Downarrow f\star_1 g"] & y
\end{tikzcd}
$$

- These compositions satisfy the **exchange law**: for every f, f', g, g' in $\mathcal{C}_2$:

$$(f \star_1 f') \star_0 (g \star_1 g') = (f \star_0 g) \star_1 (f' \star_0 g').$$

- A $\mathbb{K}$-linear 2-category is a category enriched over $\left(\mathbf{Alg}_1, \times, \begin{tikzcd} \bullet \arrow[loop above, "1"] \end{tikzcd}\right)$, where $\mathbf{Alg}_1$ is the category of linear 1-categories.
  - For every $p, q \in \mathcal{C}_1$, $\mathcal{C}(p, q)$ is a $\mathbb{K}$-vector space: for any $f, g \in \mathcal{C}(p, q)$, $\lambda f + \mu g \in \mathcal{C}_2(p, q)$.
  - $(\lambda f + \mu g) \star_0 h = \lambda f \star_0 h + \mu g \star_0 h, \quad \lambda f \star_1 \lambda h = \lambda(f \star_1 h)$.

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  - $(\lambda f + \mu g) \star_0 h = \lambda f \star_0 h + \mu g \star_0 h, \quad \lambda f \star_1 \lambda h = \lambda (f \star_1 h)$.
- The structures of ($\mathbb{K}$-linear) monoidal category and ($\mathbb{K}$-linear) 2-category with one 0-cell coincide:

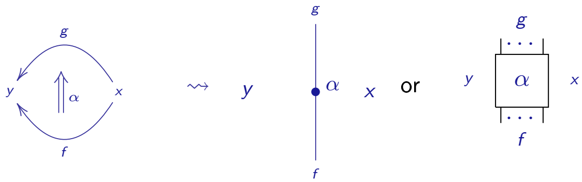
$$\text{objets de } \mathcal{A} \leftrightarrow 1\text{-cellules de } \mathcal{C}$$

$$\text{morphismes de } \mathcal{A} \leftrightarrow 2\text{-cellules de } \mathcal{C}$$

$$\otimes \leftrightarrow \star_0, \qquad \text{composition de morphismes} \leftrightarrow \star_1$$

String diagrams

- 2-cells of a ($\mathbb{K}$-linear) 2-category can be depicted using **string diagrams**, or **circuits**, as follows:



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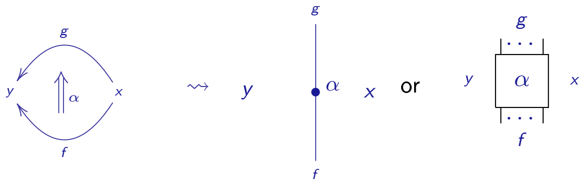

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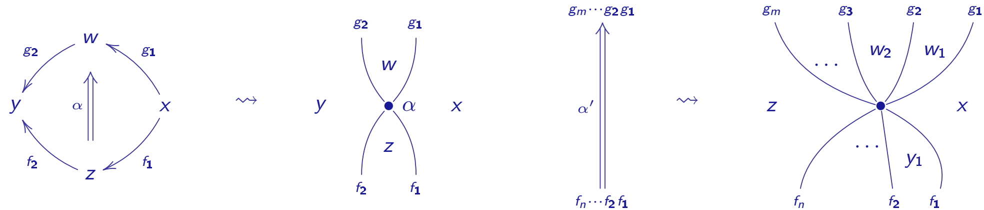

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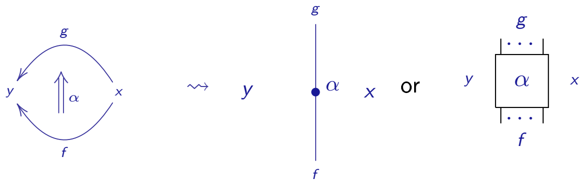

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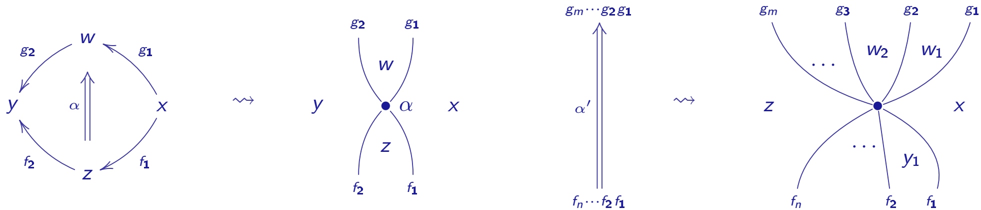
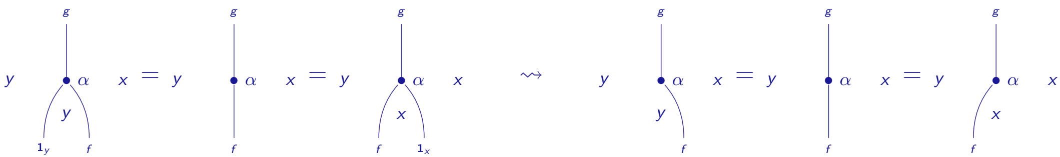


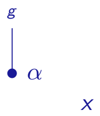
- We do not draw identity 1-cells in string diagrams:



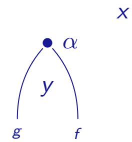
String diagrams

- **Examples:**

$$\begin{tikzcd} x & & x \arrow[ll, bend right=50, "g"'] \arrow[ll, bend left=50, "\mathbf{1}_x"] \arrow[ll, phantom, "\Uparrow\alpha"] \end{tikzcd} \rightsquigarrow$$



$$\begin{tikzcd} x & & x \arrow[ll, bend right=50, "\mathbf{1}_x"'] \arrow[dl, bend left, "f"] \arrow[ll, phantom, "\Uparrow\alpha"] \\ & y \arrow[ul, bend left, "g"] & \end{tikzcd} \rightsquigarrow$$



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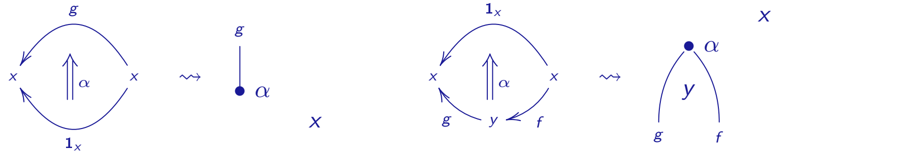
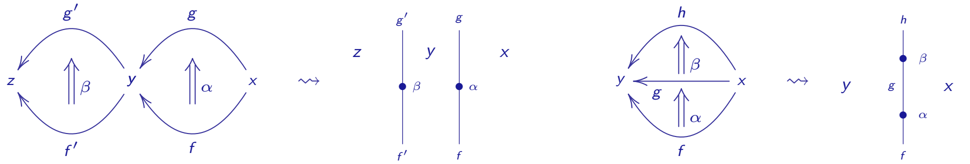


- Compositions with string diagrams:



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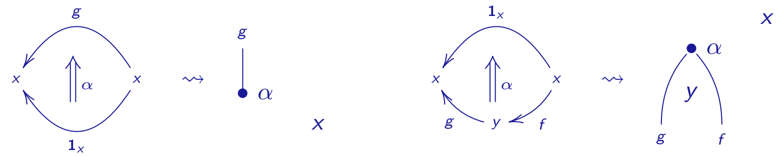

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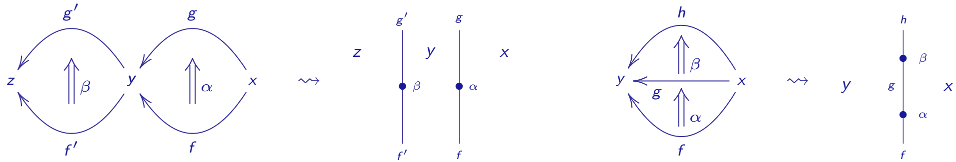
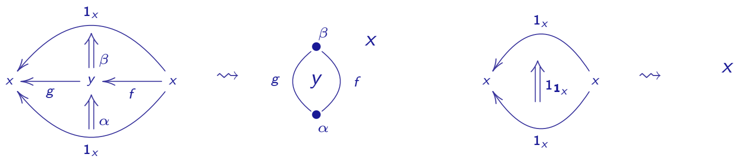
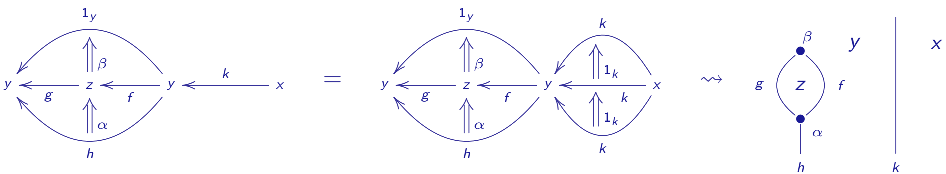


- Because identity 2-cells can be removed from composites using the identity axioms, we do not draw identity 2-cells.

- **More examples:**





String diagrams

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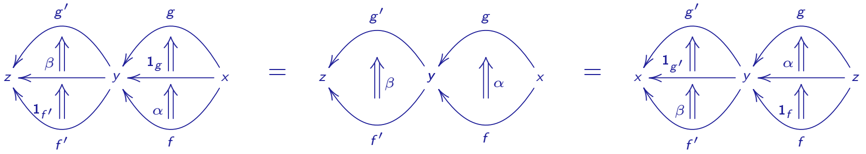

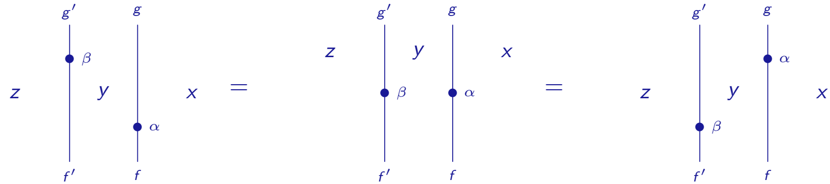

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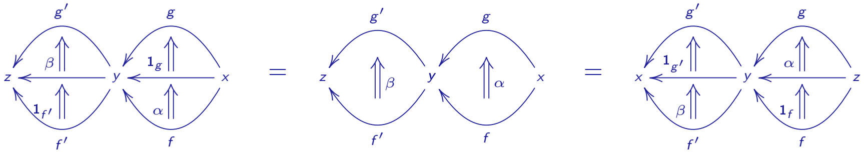
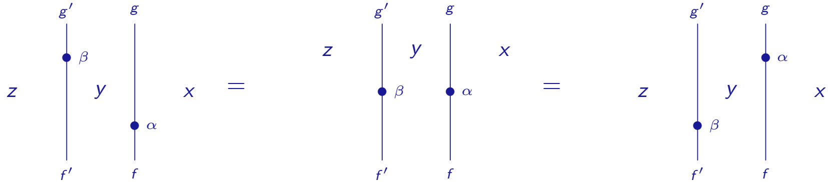
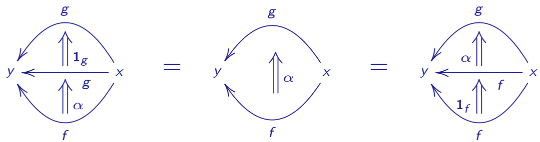
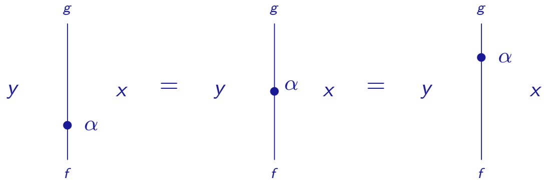


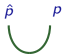


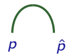
- Therefore, height of a given string diagram on a strand does not matter:

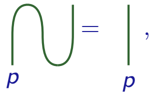


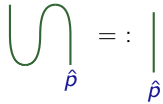


- Let $\mathcal{C}$ be a linear 2-category. If p is a 1-cell, a **left-adjoint** of p is a 1-cell $\hat{p}$ such that there are 2-cells

$\eta_p : 1 \Rightarrow p \star_0 \hat{p}, \quad \varepsilon_p : \hat{p} \star_0 p \Rightarrow 1,$ 

, 

s.t. 

, 

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Cyclic 2-cells and pivotal categories

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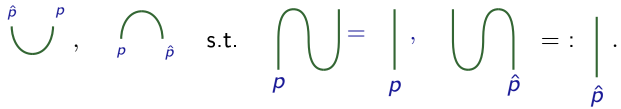

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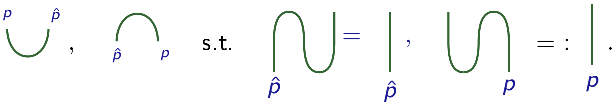

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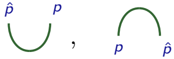

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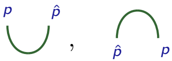

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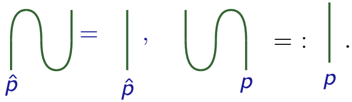
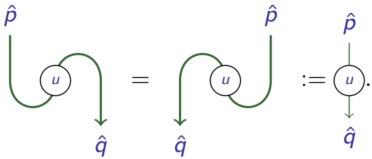


- A 2-cell $u : p \Rightarrow q$ is **cyclic** with respect to some biadjunction $(p, \hat{p})$, $(q, \hat{q})$ if



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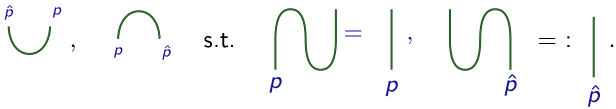

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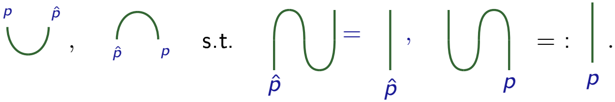

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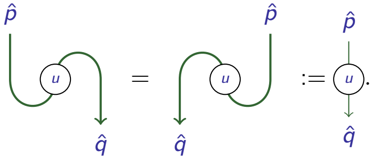

- A 2-category is **pivotal** if every 1-cell admits a left and right adjoint $\hat{p}$, and every 2-cell $f : p \Rightarrow q$ is cyclic w.r.t biadjunctions $(p, \hat{p})$, $(q, \hat{q})$.

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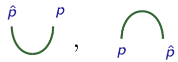
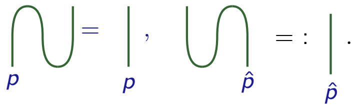
$\eta_p : 1 \Rightarrow p \star_0 \hat{p}, \quad \varepsilon_p : \hat{p} \star_0 p \Rightarrow 1,$ 

s.t. 

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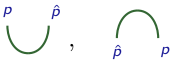
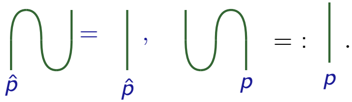


s.t. 

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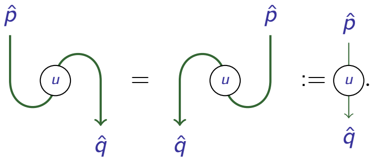

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- **Fact:** In a pivotal 2-category, two string diagrams that are equal up to isotopy represent the same 2-cell.

II. Diagrammatic rewriting and linear $(3,2)$-polygraphs

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\begin{aligned}
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\tau_i^2 &= 0 \\
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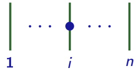
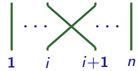
$x_i =$ 

, $\tau_i =$ 

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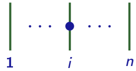
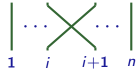
$x_i =$ 

, $\tau_i =$ 

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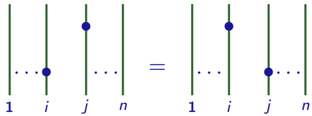

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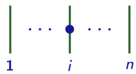
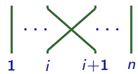
$x_i =$ 

, $\tau_i =$ 

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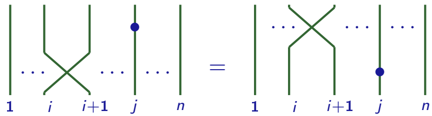

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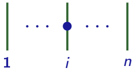
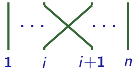
$x_i =$ 

, $\tau_i =$ 

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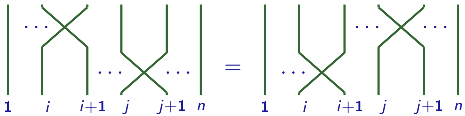

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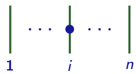
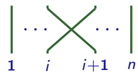
$x_i =$ 

, $\tau_i =$ 

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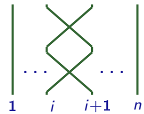

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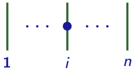
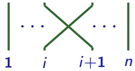
$x_i =$ 

, $\tau_i =$ 

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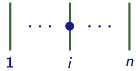
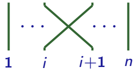
$x_i =$ 

, $\tau_i =$ 

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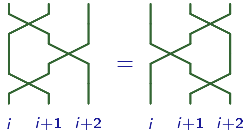

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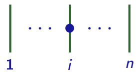
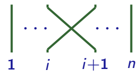
$x_i =$ 

, $\tau_i =$ 

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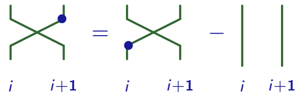

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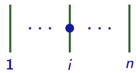
$x_i =$ 

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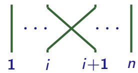
$\tau_i =$ 

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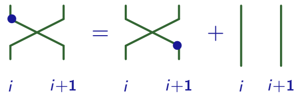

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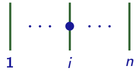
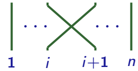
$x_i =$ 

, $\tau_i =$ 

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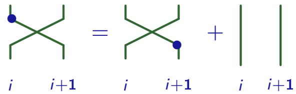

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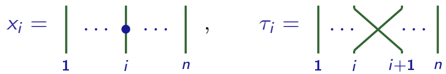

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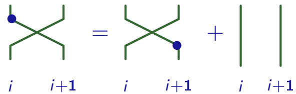

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  - $\mathcal{NH}_0 = \{\bullet\}$,
  - $\mathcal{NH}_1 = \{n \in \mathbb{N}\}$ (number of strands on sources and targets),
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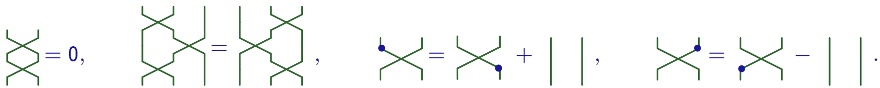

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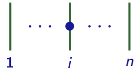
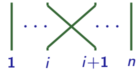
$x_i =$ 

, $\tau_i =$ 

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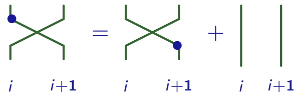

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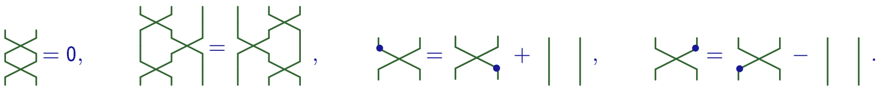

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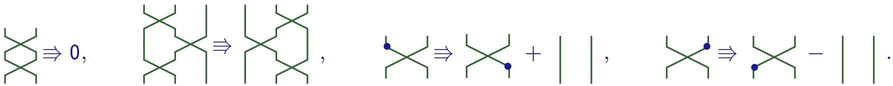
  - $P_3 = \{$  $\cdot\}$

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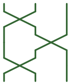
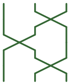
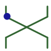
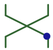
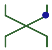
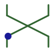
  - $P_3 = \{$  $\Rrightarrow 0,$  $\Rrightarrow$  $,$  $\Rrightarrow$  $+$  $,$  $\Rrightarrow$  $-$ $\cdot\}$
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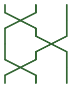
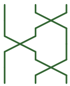
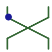
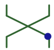
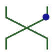
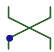
  - $P_3 = \{$  $\Rrightarrow 0,$  $\Rrightarrow$  ,  $\Rrightarrow$  $+$  ,  $\Rrightarrow$  $-$  $.\}$
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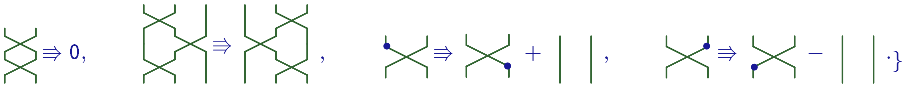
  - $P_3 = \{$ 

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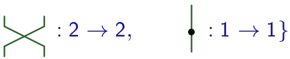
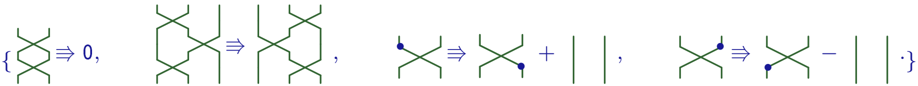
  - $P_2 = \{$  $\}$
  - $P_3 = \{$  $\}$
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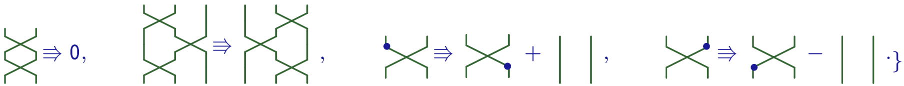
  - $P_3 = \{$  $\}$
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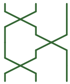
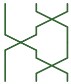
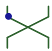
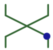
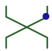
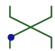
  - $P_3 = \{$  $\Rrightarrow 0,$  $\Rrightarrow$  ,  $\Rrightarrow$  $+$  ,  $\Rrightarrow$  $-$  $\cdot\}$
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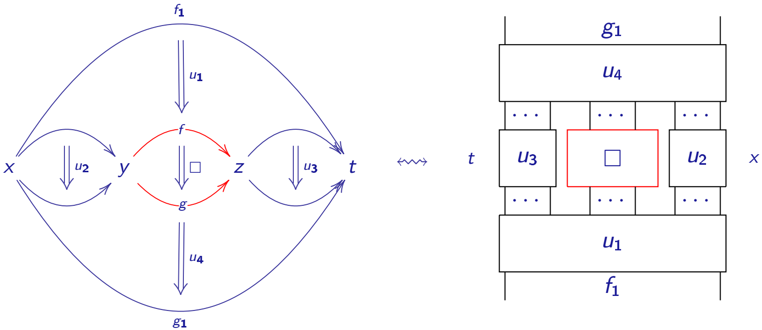

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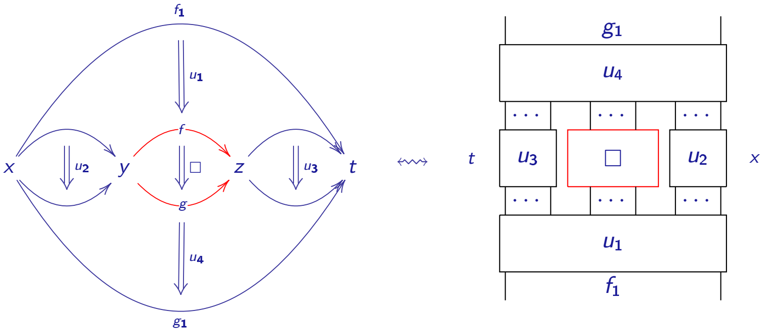

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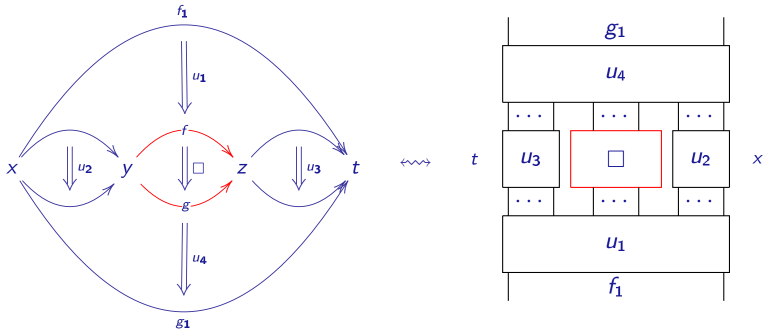

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- A **rewriting step** of P is a 3-cell of the form

$$c[\alpha] : c[s_2(\alpha)] \Rrightarrow c[t_2(\alpha)]$$

for $\alpha \in P_3$ where c is a linear context such that **the monomial** $u_1 \star_1 (u_2 \star_0 s_2(\alpha) \star_0 u_3) \star_1 u_4$ **does not appear in the polynomial** h.

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- The green condition is needed to avoid trivial non-termination: if $u \Rrightarrow v$, then we have $-u \Rrightarrow -v$, which implies

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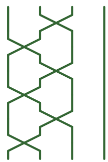

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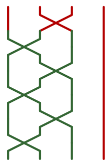

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- **Theorem [Newman lemma]:** If P is terminating, confluence and local confluence are equivalent.
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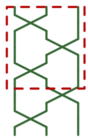
$s(\alpha)$ 

Rewriting in linear 2-categories

- The green condition is needed to avoid trivial non-termination: if $u \Rrightarrow v$, then we have $-u \Rrightarrow -v$, which implies
$$v = (u+v) - u \Rrightarrow (u+v) - v = u.$$

- **Assumption :** We consider **left-monomial** linear $(3,2)$-polygraphs, that is for every $\alpha \in P_3$, $s_2(\alpha)$ is a monomial.

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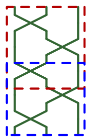

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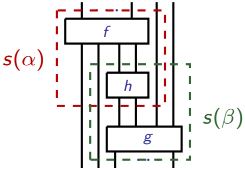

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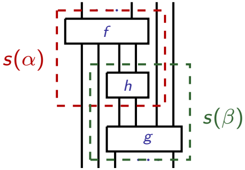

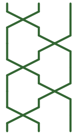

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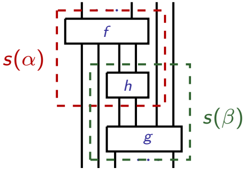

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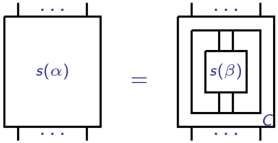

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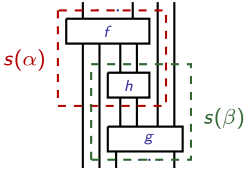

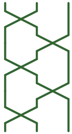

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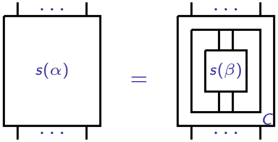

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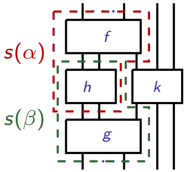

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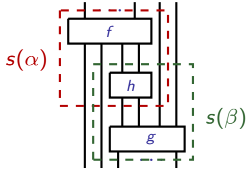

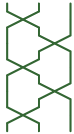

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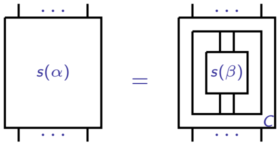

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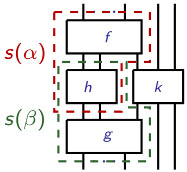

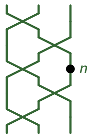

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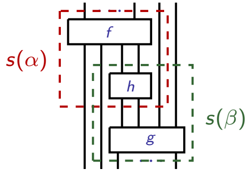

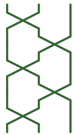

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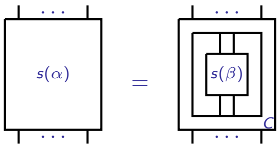

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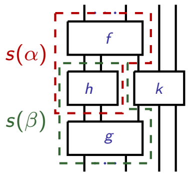

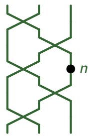

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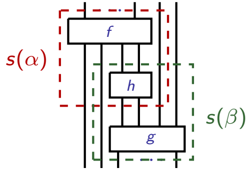

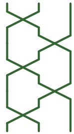

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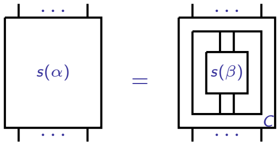

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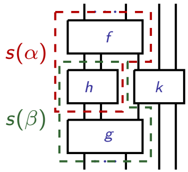

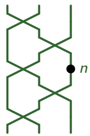

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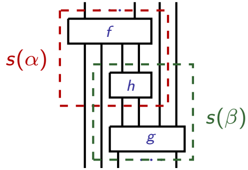

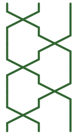

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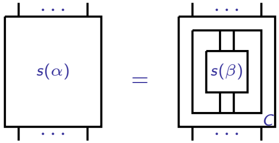

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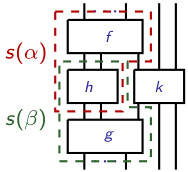

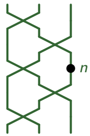

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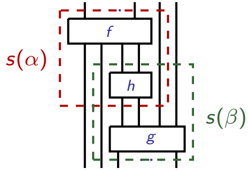

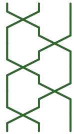

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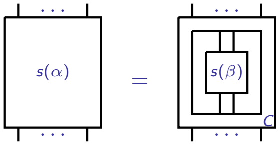

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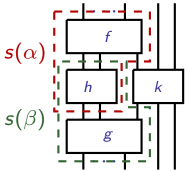

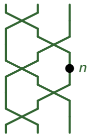

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- We need to find criteria to prove termination of linear $(3,2)$-polygraphs, and then confluence is a check of critical branchings.

Termination of 3-polygraphs

- In order to prove termination, we often want to define well-founded total orders satisfying:
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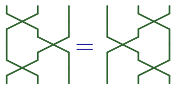

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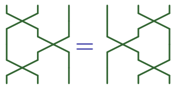

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- The correct setting to define these orders is the one of **derivations**, as introduced by **Guiraud '04** and **Guiraud-Malbos '09**.

Termination using derivations

- **Idea of the construction:**
  - Each 2-cell is seen as an electronical circuit whose components are given by the generating 2-cells.
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$$f \prec g \quad \text{iff} \quad d(f) < d(h) \text{ for any monomial } h \text{ in } g.$$

- Given a 2-category $\mathcal{C}$, define the **category of contexts** $\mathbf{C}[\mathcal{C}]$ of $\mathcal{C}$ by:
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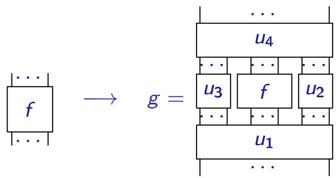

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- **Idea of the construction:**
  - Each 2-cell is seen as an electronical circuit whose components are given by the generating 2-cells.
  - Each generator will receive a certain intensity of ascending and descending currents, calculated with mappings X and Y on generators, extended functorially.
  - Any 2-cell will produce some heat, which is calculated by the sum of heats produced by any of its components, depending on they intensity of current they receive.
  - Heats are calculated using derivations d.
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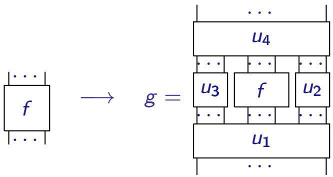

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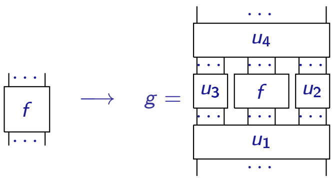

- A $\mathcal{C}$-module is a functor $M : \mathbf{C}[\mathcal{C}] \to \mathbf{Ab}$, where $\mathbf{Ab}$ is the category of abelian groups.

Modules for 2-categories

- **Exemple:** In the case of 2-categories, we construct prototypical modules. Let **Ord** be the 2-category with one 0-cell, 1-cells are partially ordered sets, and 2-cells are monotone maps.
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\end{tikzcd}
\longrightarrow
\begin{tikzcd}
y \arrow[r, bend left=80, "p'"{name=P1}] \arrow[r, bend left, "p"{name=P}] \arrow[r, bend right, "q"'{name=Q}] \arrow[r, bend right=80, "q'"'{name=Q1}] \arrow[from=P1, to=P, phantom, "\Downarrow u"] \arrow[from=P, to=Q, phantom, "\Downarrow v"] \arrow[from=Q, to=Q1, phantom, "\Downarrow w"] & z
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- When $\mathcal{C} = P_2^*$ is freely generated by a 2-polygraph, such a $\mathcal{C}$-module is uniquely determined by $X(p)$ and $Y(p)$ for $p \in P_1$ and morphisms $X(u) : X(p) \to X(q)$ and $Y(u) : Y(q) \to Y(p)$ for every $u : p \Rightarrow q \in P_2$.

Termination using derivations

- A **derivation** of a 2-category C into a $\mathcal{C}$-module M is a map sending every 2-cell u in $\mathcal{C}$ to an element $d(u) \in M(u)$ such that

$$d(u \star_i v) = u \star_i d(v) + d(u) \star_i v,$$

where $u \star_i d(v) = M(u \star_i \Box)(d(v))$ and $d(u) \star_i v = M(\Box \star_i v)(d(u))$.

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  - Two 2-functors $X : P_2^* \to \mathbf{Ord}$ and $Y : (P_2^*)^{\mathrm{op}} \to \mathbf{Ord}$ such that for every 1-cell p in P_1, the sets $X(p)$ and $Y(p)$ are non-empty and for every generating 3-cell α in P_3, the inequalities $X(s_2(\alpha)) \geq X(h)$ and $Y(s_2(\alpha)) \geq Y(h)$ hold for every non identity monomial h in $t_2(\alpha)$.
  - An abelian group G in $\mathbf{Ord}$ whose addition is strictly monotone in both arguments and such that every decreasing sequence of non-negative elements of G is stationary.
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  Then the linear $(3,2)$-polygraph P terminates.

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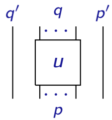
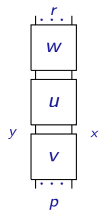
- In general, we consider $G = \mathbb{Z}$ and Y to be the trivial 2-functor, that is $Y(p) = \varnothing$ for any $p \in P_1$, and $Y(u)$ is the trivial map $Y(q) \Rightarrow Y(p)$ for $u : p \Rightarrow q \in P_2$.
  - One might forget about the Y in the definition of $M_{X,Y,G}$:





$$a \mapsto : \ (X(p') \times X(p) \times X(q') \to G, (x', x, x'') \mapsto a(x)) \qquad a \mapsto (X(p') \to G, (x, y) \mapsto a(X(p')(x)))\,.$$

Termination using derivations: an example

- **Steps of derivation:** Let $P = (P_0, P_1, P_2, P_3)$ be a linear $(3,2)$-polygraph with $P_3 = A \sqcup B$, functors X, Y and a derivation d such that

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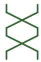
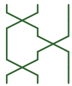
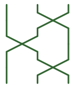
- **Example:** Consider the linear $(3,2)$-polygraph of **permutations** P_{Sym} defined by $P_0 = \{\bullet\}$, $P_1 = \{1\}$,

$P_2 = \{$  $\}$, $\qquad P_3 = \{$  $\overset{A}{\Rrightarrow} 0$,  $\overset{B}{\Rrightarrow}$  $.\}$

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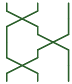
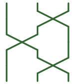
  $P_2 = \{$  $\}$, $\qquad P_3 = \{$  $\overset{A}{\Rrightarrow} 0$,  $\overset{B}{\Rrightarrow}$  $\cdot\}$

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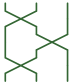
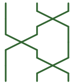
  $P_2 = \{$  $\}$, $\qquad P_3 = \{$  $\overset{A}{\Rrightarrow} 0$,  $\overset{B}{\Rrightarrow}$  $.\}$

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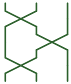
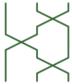
  $P_2 = \{$  $\}$, $\qquad P_3 = \{$  $\overset{A}{\Rrightarrow} 0$,  $\overset{B}{\Rrightarrow}$  $\cdot\}$

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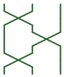
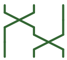
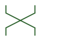
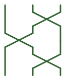
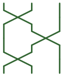
  $X($  $)(n,m,k) = X($  $)(n,k,m+1) = X($  $)(k, n+1, m+1)$

  $= (k, m+1, n+2) = X($  $)(n,m,k)$

- Recall that

$X($  $)(n, m) = (m, n+1) \quad d($  $)(n, m) = m.$

- Compute the values of $d(s_2(B))$ and $d(t_2(B))$:

$d($  $)(n, m, k)$

Termination using derivations: an example

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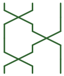
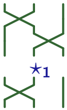
$d($  $)(n, m, k) = d($ 

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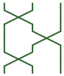
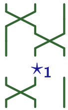
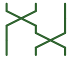
$d($  $)(n, m, k) = d($  $)(n, m, k) = d($  $)(n, m, k) + d($  $)(X \Big($  $\Big)(n, m, k)$

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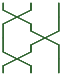
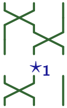
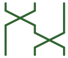
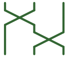
$d($  $)(n, m, k) = d($  $)(n, m, k) = d($  $)(n, m, k) + d($  $)(X\big($  $\big)(n, m, k)$

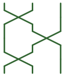
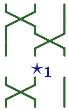
$= d($  $)(m, k) + d($  $)(n, k, m+1)$

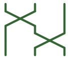
Termination using derivations: an example

- Recall that

$X($  $)(n, m) = (m, n + 1)$ $d($  $)(n, m) = m.$

- Compute the values of $d(s_2(B))$ and $d(t_2(B))$:

$d($  $)(n, m, k) = d($  $)(n, m, k) = d($  $)(n, m, k) + d($  $)(X($  $)(n, m, k))$

$= d($  $)(m, k) + d($  $)(n, k, m + 1)$

$= k + d($  $)(n, k, m + 1) + d($  $)(k, n + 1, m + 1)$

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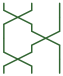
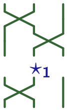
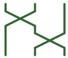
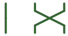
$d($  $)(n, m, k) = d($ 

$)(n, m, k) = d($  $)(n, m, k) + d($  $)(X\Big($  $\Big)(n, m, k)$

$= d($  $)(m, k) + d($  $)(n, k, m+1)$

$= k + d($  $)(n, k, m+1) + d($  $)(k, n+1, m+1)$

$= 2k + m + 1$

Termination using derivations: an example

- Recall that
$$X(\,)(n,m) = (m, n+1) \quad d(\,)(n,m) = m.$$

- Compute the values of $d(s_2(B))$ and $d(t_2(B))$:
$$d(\,)(n,m,k) = d(\ \star_1\ )(n,m,k) = d(\,)(n,m,k) + d(\,)(X(\,))(n,m,k)$$
$$= d(\,)(m,k) + d(\,)(n,k,m+1)$$
$$= k + d(\,)(n,k,m+1) + d(\,)(k,n+1,m+1)$$
$$= 2k + m + 1$$

- Similarly,
$$d(\,) = d(\,)(n,m,k) + d(\,)(m,n+1,k) + d(\,)(m,k,n+2)$$
$$= m + 2k.$$
$$d(\,)(n,m) = d(\,)(n,m) + d(\,)(m,n+1) = n+m+1.$$

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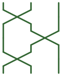
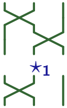
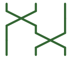
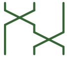
$d($  $)(n, m, k) = d($  $)(n, m, k) = d($  $)(n, m, k) + d($  $)(X($  $)(n, m, k))$

$= d($  $)(m, k) + d($  $)(n, k, m+1)$

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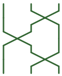
$= 2k + m + 1$

- Similarly,

$d($  $) = d($  $)(n, m, k) + d($  $)(m, n+1, k) + d($  $)(m, k, n+2)$

$= m + 2k.$

$d($  $)(n, m) = d($  $)(n, m) + d($  $)(m, n+1) = n + m + 1.$

- Therefore, X and d satisfy the required conditions, and the linear $(3, 2)$-polygraph of permutations is terminating.

Example: Khovanov-Lauda-Rouquier (KLR) algebras

- These algebras appear in the process of categorifying a quantum groupe $\mathbf{U}_q(\mathfrak{g})$ associated with a symmetrizable Kac-Moody algebra $\mathfrak{g}$.

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- For such an element $\mathcal{V}$, we define an algebra $R(\mathcal{V})$.

- **Theorem [Khovanov-Lauda '08]:** If $R = \bigoplus_{\mathcal{V} \in \mathbb{N}[I]} R(\mathcal{V})$,

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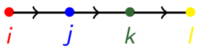
  $\Gamma =$  (Γ simply laced)

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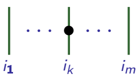
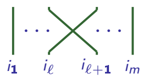
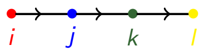
- $R(\mathcal{V})$ is generated by

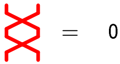
  $x_{k,\mathbf{i}} =$  and $\tau_{k,\mathbf{i}} =$ 

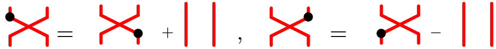
  for any $\mathbf{i} = i_1 \dots i_m \in \mathrm{Seq}(\mathcal{V})$, $1 \leq k \leq m$ and $1 \leq \ell < m$.

Convergent presentation of the KLR algebras

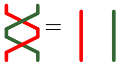
- Relations to realize the algebras $R(\mathcal{V})$ as 2Hom-spaces of a linear 2-category: ($\Gamma =$ )

i) Same color:

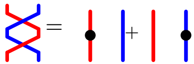




ii) Distant colors:

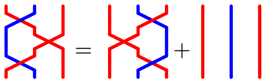


iii) Close colors:

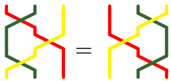


iv) Different colors:

vi) Braid relations:



and otherwise



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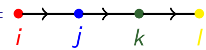
- Relations to realize the algebras $R(\mathcal{V})$ as 2Hom-spaces of a linear 2-category: ($\Gamma =$ )

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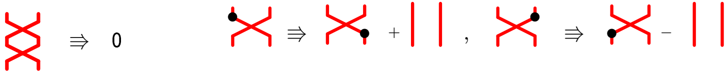

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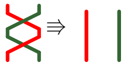

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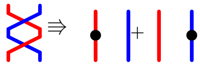

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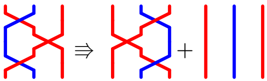

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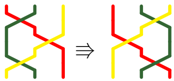

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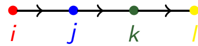
- Relations to realize the algebras $R(\mathcal{V})$ as 2Hom-spaces of a linear 2-category: ($\Gamma =$ )

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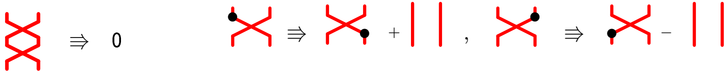

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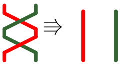

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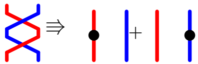

iv) Different colors:

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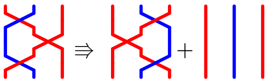
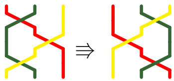
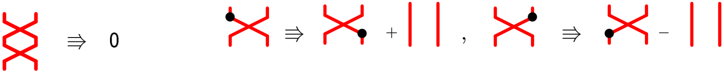
 and otherwise 

- **Theorem [D. '19]:** This linear $(3,2)$-polygraph is convergent.

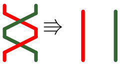
Convergent presentation of the KLR algebras

- Relations to realize the algebras $R(\mathcal{V})$ as 2Hom-spaces of a linear 2-category: ($\Gamma = i \to j \to k \to l$)

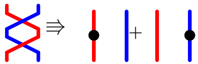
  i) Same color:



  ii) Distant colors:

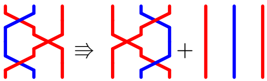


  iii) Close colors:

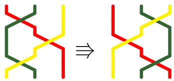


  iv) Different colors:

  vi) Braid relations:



and otherwise



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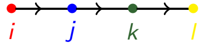
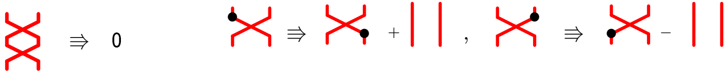
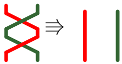
Convergent presentation of the KLR algebras

- Relations to realize the algebras $R(\mathcal{V})$ as 2Hom-spaces of a linear 2-category: ($\Gamma =$ )

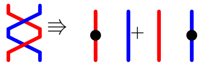
  i) Same color:



  ii) Distant colors:



  iii) Close colors:



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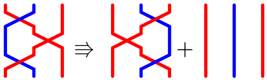
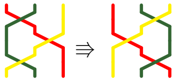


and otherwise



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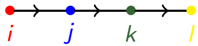
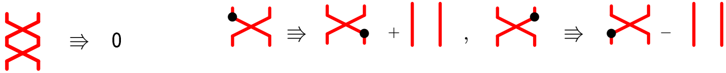
Convergent presentation of the KLR algebras

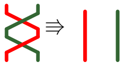
- Relations to realize the algebras $R(\mathcal{V})$ as 2Hom-spaces of a linear 2-category: ($\Gamma =$ 

)

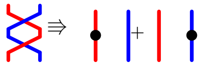
i) Same color:



ii) Distant colors:



iii) Close colors:



iv) Different colors:

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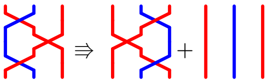
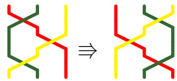
 and otherwise 

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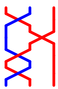
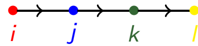


Convergent presentation of the KLR algebras

- Relations to realize the algebras $R(\mathcal{V})$ as 2Hom-spaces of a linear 2-category: ($\Gamma =$ 

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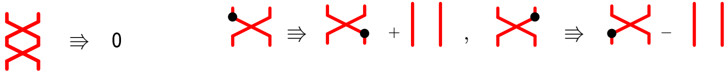
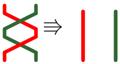
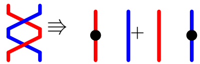


  ii) Distant colors:



  iii) Close colors:



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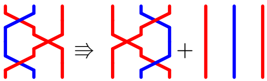

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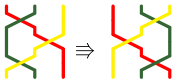

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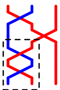
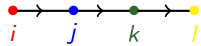
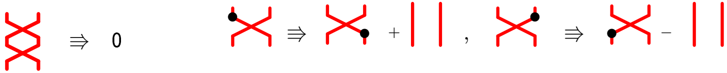


Convergent presentation of the KLR algebras

- Relations to realize the algebras $R(\mathcal{V})$ as 2Hom-spaces of a linear 2-category: ($\Gamma =$ )

i) Same color:



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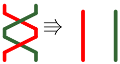

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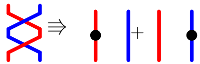

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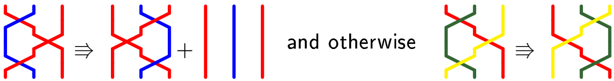

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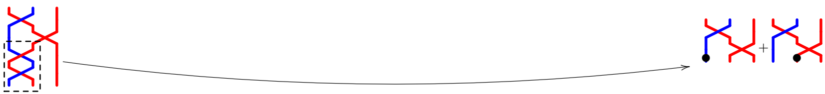
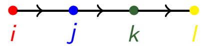
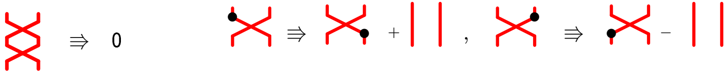


Convergent presentation of the KLR algebras

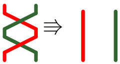
- Relations to realize the algebras $R(\mathcal{V})$ as 2Hom-spaces of a linear 2-category: ($\Gamma =$ 

)

  i) Same color:

  ii) Distant colors:

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  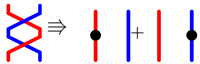

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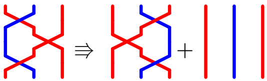
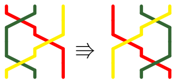
   and otherwise 

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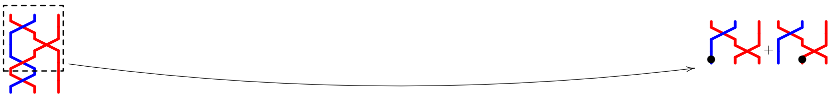
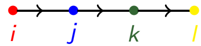


Convergent presentation of the KLR algebras

- Relations to realize the algebras $R(\mathcal{V})$ as 2Hom-spaces of a linear 2-category: ($\Gamma =$ 

)

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  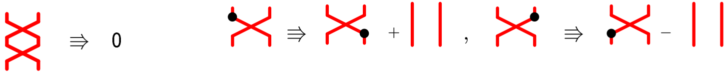

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  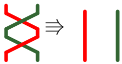

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  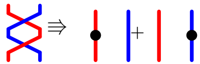

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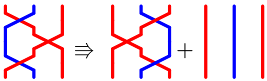
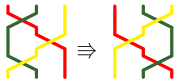
   and otherwise 

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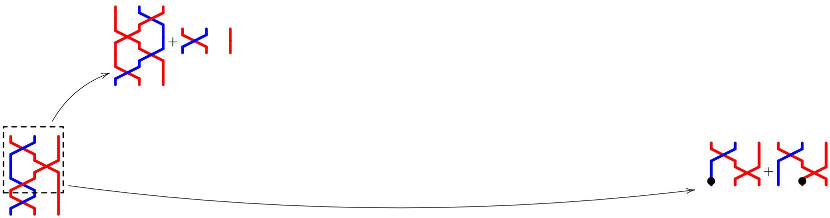
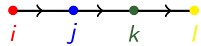


Convergent presentation of the KLR algebras

- Relations to realize the algebras $R(\mathcal{V})$ as 2Hom-spaces of a linear 2-category: ($\Gamma =$ )

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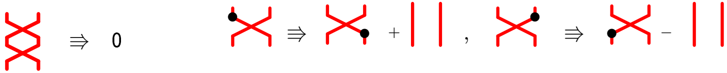

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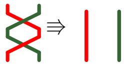

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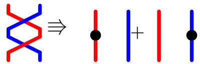

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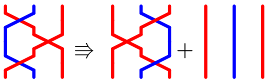
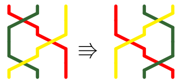
 and otherwise 

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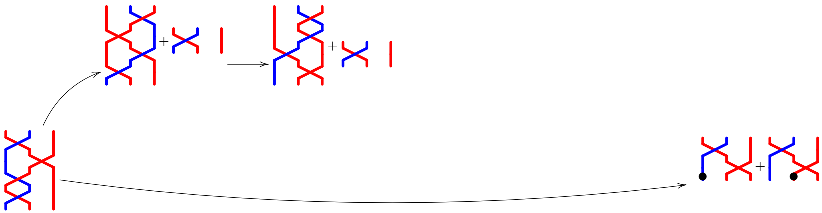

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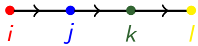
- Relations to realize the algebras $R(\mathcal{V})$ as 2Hom-spaces of a linear 2-category: ($\Gamma =$ )

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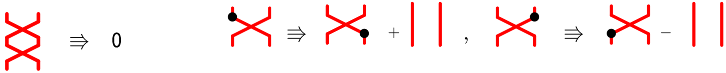

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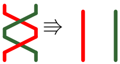

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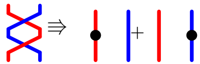

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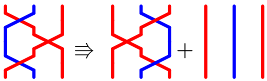
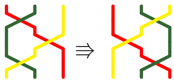
 and otherwise 

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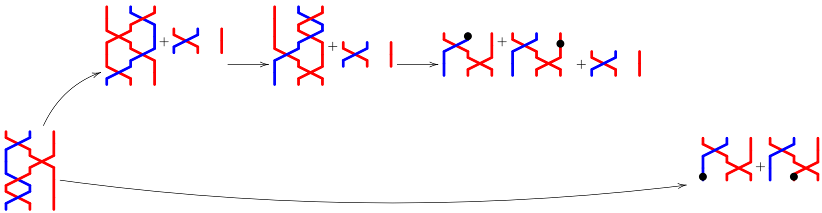

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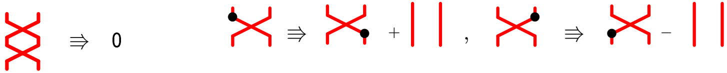

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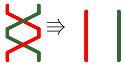

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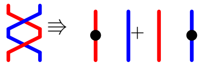
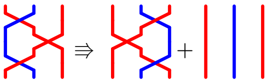
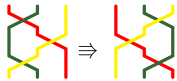


  iv) Different colors:

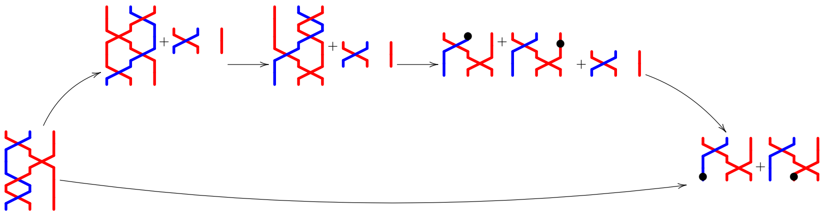
  vi) Braid relations:



and otherwise



- **Theorem [D. '19]:** This linear $(3,2)$-polygraph is convergent.
  - Termination: use derivations in two steps, values on generators are independent of the colors.
  - Confluence: exhaustive study of all critical branchings.



III. Categorification of the quantum group $U_q(\mathfrak{sl}_2)$

Following Aaron Lauda: *An introduction to diagrammatic algebra and categorified quantum* $\mathfrak{sl}_2$

The Lie algebra $\mathfrak{sl}_2$

- A **Lie algebra** over a field $\mathbb{K}$ is a $\mathbb{K}$-vector space $\mathfrak{g}$ equipped with a Lie bracket $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisying:
  - bilinearity: $[\lambda x + \mu x', y] = \lambda[x, y] + \mu[x', y], \quad [x, \delta y + \gamma y'] = \delta[x, y] + \gamma[x, y']$.
  - antisymmetry: $[x, y] = -[y, x]$.
  - Jacobi identity: $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$.

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- A **representation** of a Lie algebra $\mathfrak{g}$ is a $\mathbb{K}$-vector space V with a Lie algebra morphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V) := \mathrm{End}(V)$, that is

$$\rho([x,y]) = \rho(x) \circ \rho(y) - \rho(y) \circ \rho(x)$$

Notation: $x \cdot v := \rho(x)(v)$.

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- **Example:** The lie algebra $\mathfrak{sl}_2$ of 2×2 traceless matrices: $\mathfrak{sl}_2 = \mathbb{K}e \oplus \mathbb{K}h \oplus \mathbb{K}f$ where

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

  and satisfying $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$.

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  and satisfying $[h,e] = 2e$, $[h,f] = -2f$, $[e,f] = h$.
- Let V be a finite dimensional representation of $\mathfrak{sl}_2$. It can be decomposed as $V = \bigoplus V_\alpha$, where

$$V_\alpha = \{v \in \mathbb{K}; h \cdot v = \alpha v\}.$$

  - Action of e and f on V_α:
    $h(e(v)) = e(h(v)) + [h,e](v) = e(\alpha v) + 2e(v) = (\alpha+2)e(v)$, and similarly, $h(f(w)) = (\alpha - 2)f(w)$.
  - Therefore, $e : E : V_\alpha \to V_{\alpha+2}$ and $f : V_\alpha \to V_{\alpha-2}$.
  - An irreducible representation V admits a decomposition $V = \bigoplus_{n\in\mathbb{Z}} V_n$ where V_n is called the n-th weight space, and $\mathbb{Z}$ is the **weight lattice** of $\mathfrak{sl}_2$.

The quantum group $\mathbf{U}_q(\mathfrak{sl}_2)$

- The quantum group $\mathbf{U}_q(\mathfrak{sl}_2)$ associated with $\mathfrak{sl}_2$ is the $\mathbb{Q}(q)$-algebra generated by elements E, F, K, K^{-1} subject to relations

$$KE = q^2 EK, \qquad KF = q^{-2} FK,$$

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- Similarly, a representation of $\mathbf{U}_q(\mathfrak{sl}_2)$ can be decomposed as $V = \bigoplus_{n\in\mathbb{Z}} V_n$ where

$$V_n = \{v \in \mathbb{K}; h \cdot v = q^n v\}.$$

- Given a weight vector $v \in V_n$ the weights of Ev and Fv are determined using the relations

$$K(Ev) = q^2 EKv = q^{n+2}(Ev), \qquad K(Fv) = q^{-2} FKv = q^{n-2}(Fv),$$

so that $E\colon V_n \to V_{n+2}$ and $F\colon V_n \to V_{n-2}$.

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  so that $E\colon V_n \to V_{n+2}$ and $F\colon V_n \to V_{n-2}$.

- V can be thought of as a collection of vector spaces V_n for $n \in \mathbb{Z}$ such that:

$$\begin{tikzcd} \cdots & V_{-N} \arrow[r, bend left, "E"] & \cdots \arrow[l, bend left, "F"] & V_{n-2} \arrow[r, bend left, "E"] & V_n \arrow[l, bend left, "F"] \arrow[r, bend left, "E"] & V_{n+2} \arrow[l, bend left, "F"] & \cdots \arrow[r, bend left, "E"] & V_N \arrow[l, bend left, "F"] & \cdots \end{tikzcd}$$

  and the main $\mathbf{U}_q(\mathfrak{sl}_2)$ relation $EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$ holds.

- For $v \in V_n$, we thus have

$$(EF - FE)v = \frac{K - K^{-1}}{q - q^{-1}} v = \frac{Kv - K^{-1}v}{q - q^{-1}} = \frac{q^n - q^{-n}}{q - q^{-1}} v = [n]v.$$

  where $[n] := \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + q^{1-n}$ is called the **quantum number n**.

Lusztig's completion

- We will consider a modified form $\dot{\mathbf{U}}$ of $\mathbf{U}_q(\mathfrak{sl}_2)$ that is better suited to study representations, and that can be interpreted as a 1-category, so that it is more natural for categorification purposes.

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- **First step:** Consider the $\mathbb{Q}(q)$-algebra obtained from $\mathbf{U}_q(\mathfrak{sl}_2)$ by adding a collection of orthogonal idempotents 1_n for $n \in \mathbb{Z}$:

$$1_n 1_m = \delta_{n,m} 1_m, \quad K1_n = 1_n K = q^n 1_n, \quad E1_n = 1_{n+2}E, \quad F1_n = 1_{n-2}F.$$

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- **Second step:** Consider an integral form of this algebra, that is a $\mathbb{Z}[q, q^{-1}]$-algebra $\dot{\mathbf{U}}$ generated by $K, K^{-1}, E^{(a)}, F^{(b)}$ for $a, b \in \mathbb{Z}_+$ where $E^{(a)}$ and $F^{(b)}$ are divided powers defined by

$$E^{(a)} = \frac{E^a}{[a]!}, \quad F^{(b)} := \frac{F^b}{[b]!} \quad \text{with } [a]! = [a][a-1]\dots[1].$$

- The idempotents $(1_n)_{n \in \mathbb{Z}}$ in $\dot{\mathbf{U}}$ satisfy

$$K 1_n = 1_n K = q^n 1_n, \quad E^{(a)} 1_n = 1_{n+2a} E^{(a)}, \quad F^{(a)} 1_n = 1_{n-2a} F^{(a)}.$$

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- We have a direct sum decomposition $\dot{\mathbf{U}} = \bigoplus_{n,m\in\mathbb{Z}} 1_m \dot{\mathbf{U}} 1_n$ where $1_m \dot{\mathbf{U}} 1_n$ is the $\mathbb{Z}[q, q^{-1}]$-algebra generated by products $1_m E^{(a)} F^{(b)} 1_n$ and $1_m F^{(b)} E^{(a)} 1_n$ for all $a, b \in \mathbb{Z}_+$ such that $n + 2a - 2b = m$.

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- $\dot{\mathbf{U}}$ can be interpreted as a 1-category whose:
  - objects are the $n \in \mathbb{Z}$,
  - morphisms with source n and target m are the elements of $1_m \dot{\mathbf{U}} 1_n$,
  - identities are the 1_n and composition $1_{m'} \dot{\mathbf{U}} 1_m \circ 1_{n'} \dot{\mathbf{U}} 1_n \to 1_{m'} \circ \dot{\mathbf{U}} 1_n$ is defined if $n' = m$ and corresponds to the product.

Categorification of $\dot{\mathbf{U}}$: 0-cells and 1-cells

- $\dot{\mathbf{U}}$ being a 1-category, we expect its categorification $\mathcal{U}(\mathfrak{sl}_2)$ to be an additive 2-category, that is a category enriched over the category of additive categories:

$$\begin{tikzcd} \mathcal{U}(\mathfrak{sl}_2) \arrow[d, "K_0"'] \\ \dot{\mathbf{U}} \end{tikzcd}$$

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- **Fact:** $\dot{\mathbf{U}}$ admits a basis $\mathbb{B}$ with good properties, called **Lusztig's canonical basis** made of elements $E^{(a)}1_{-n}F^{(b)}$ and $F^{(b)}1_nE^{(a)}$ with $a, b, n \in \mathbb{N}$ and $n \geq a + b$.
    - The structure coefficients are this basis are in $\mathbb{N}[q, q^{-1}]$.
    - This suggests that the indecomposable 1-morphisms in $\mathcal{U}(\mathfrak{sl}_2)$ should correspond (up to grading shift) to the elements of $\mathcal{B}$:

$$[b_x][b_y] = \sum_z m^z_{x,y}[b_z] \qquad \leftrightsquigarrow \qquad b_x \star_1 b_y = \bigoplus_z \bigoplus_{m^z_{x,y}} b_z$$

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  - We lift the elements 1_n, $E1_n$ and $F1_n$ of $\dot{\mathbf{U}}$ as generating 1-cells $\mathbf{1}_n$, $\mathcal{E}\mathbf{1}_n$ and $\mathcal{F}\mathbf{1}_n$ with

$$\mathbf{1}_n : n \to n, \quad \mathbf{1}_{n+2}\mathcal{E}\mathbf{1}_n : n \to n+2, \quad \mathbf{1}_{n-2}\mathcal{F}\mathbf{1}_n : n \to n-2.$$

  We simply denote $\mathbf{1}_{n+2}\mathcal{E}\mathbf{1}_n$ by $\mathcal{E}\mathbf{1}_n$, and $\mathbf{1}_n\mathcal{E}\mathbf{1}_{n-2} \circ \mathbf{1}_{n-2}\mathcal{F}\mathbf{1}_n$ is denoted by $\mathcal{E}\mathcal{F}\mathbf{1}_n$.

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$$\begin{tikzcd} \mathcal{U}(\mathfrak{sl}_2) \arrow[d, "K_0"'] & & {}_m\mathcal{U}_n \arrow[d, bend left, "K_0"] \\ \dot{\mathbf{U}} \arrow[rr, equal] & & \bigoplus\limits_{n,m\in\mathbb{Z}} 1_m\dot{\mathbf{U}}1_n \arrow[u, dotted] \end{tikzcd}$$

where $K_0(\mathcal{U}(\mathfrak{sl}_2)) := \bigoplus_{n,m\in\mathbb{Z}} K_0({}_m\mathcal{U}_n)$ with $[x] = [x_1][x_2]$ if $x = x_1 \star_0 x_2$ in $\mathcal{U}(\mathfrak{sl}_2)$

- $1_m\dot{\mathbf{U}}1_n$ is a $\mathbb{Z}[q, q^{-1}]$-module, and we lift the action of q as a grading automorphism $\{\cdot\} : \mathcal{U} \to \mathcal{U}$:

$$[x\{t\}] = q^t[x] \qquad \rightsquigarrow \text{ hom-sets } {}_m\mathcal{U}_n \text{ for } m, n \in \mathbb{Z} \text{ are graded}$$

- **Fact:** $\dot{\mathbf{U}}$ admits a basis $\mathbb{B}$ with good properties, called **Lusztig's canonical basis** made of elements $E^{(a)}1_{-n}F^{(b)}$ and $F^{(b)}1_nE^{(a)}$ with $a, b, n \in \mathbb{N}$ and $n \geq a + b$.
  - The structure coefficients are this basis are in $\mathbb{N}[q, q^{-1}]$.
  - This suggests that the indecomposable 1-morphisms in $\mathcal{U}(\mathfrak{sl}_2)$ should correspond (up to grading shift) to the elements of $\mathcal{B}$:

$$[b_x][b_y] = \sum_z m^z_{x,y}[b_z] \qquad \leftrightsquigarrow \qquad b_x \star_1 b_y = \bigoplus_z \bigoplus_{m^z_{x,y}} b_z$$

  - We lift the elements 1_n, $E1_n$ and $F1_n$ of $\dot{\mathbf{U}}$ as generating 1-cells $\mathbf{1}_n$, $\mathcal{E}\mathbf{1}_n$ and $\mathcal{F}\mathbf{1}_n$ with

$$\mathbf{1}_n : n \to n, \quad \mathbf{1}_{n+2}\mathcal{E}\mathbf{1}_n : n \to n+2, \quad \mathbf{1}_{n-2}\mathcal{F}\mathbf{1}_n : n \to n-2.$$

    We simply denote $\mathbf{1}_{n+2}\mathcal{E}\mathbf{1}_n$ by $\mathcal{E}\mathbf{1}_n$, and $\mathbf{1}_n\mathcal{E}\mathbf{1}_{n-2} \circ \mathbf{1}_{n-2}\mathcal{F}\mathbf{1}_n$ is denoted by $\mathcal{E}\mathcal{F}\mathbf{1}_n$.

  - Therefore, the 1-cells of $\mathcal{U}(\mathfrak{sl}_2)$ with source n and target m are formal direct sums of elements of the form

$$\mathbf{1}_m\mathcal{E}^{\alpha_1}\mathcal{F}^{\beta_1}\mathcal{E}^{\alpha_2}\ldots\mathcal{F}^{\beta_{k-1}}\mathcal{E}^{\alpha_k}\mathcal{F}^{\beta_k}\mathbf{1}_n\{s\}$$

    where $m = n + 2(\sum\alpha_i - \sum\beta_i)$, and $s \in \mathbb{Z}$.

Categorification of $\dot{\mathbf{U}}$: 2-cells

- 0-cells of $\mathcal{U}(\mathfrak{sl}_2)$ are elements of the weight lattice $\mathbb{Z}$ of $\mathfrak{sl}_2$, and 1-cells of $\mathcal{U}(\mathfrak{sl}_2)$ are formal direct sums of elements of the form

$$\mathbf{1}_{n'}\mathcal{E}_{\underline{\varepsilon}}\mathbf{1}_n\{t\}$$

where $\underline{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_k)$, each $\varepsilon_i \in \{+,-\}$, $\mathcal{E}_+ := \mathcal{E}$, $\mathcal{E}_- := \mathcal{F}$ and $n' - n = 2\sum_{i=1}^{k} \varepsilon_i 1$.

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- We thus get a pairing $\langle [x], [y] \rangle$ for $x, y \in \mathcal{U}(\mathfrak{sl}_2)_1$ such that

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  - The graded HOMs $\mathcal{U}(\mathfrak{sl}_2)(x, y)$ categorify a semi-linear form on $\dot{\mathbf{U}}$.
  - **Candidate :** Lusztig defined such a pairing on $\dot{\mathbf{U}}$ as the dimension of an Ext algebra between sheaves over a quiver variety. It satisfies

$$\langle\cdot,\cdot\rangle \text{ is semi-linear, and } \langle 1_{n_1} x 1_{n_2}, 1_{n'_1} y 1_{n'_2}\rangle = 0 \text{ for every } x, y, \text{ unless } n_1 = n'_1,\ n_2 = n'_2.$$

  - Moreover, one can compute values of this pairing on products of divided powers.

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- We define identity 2-cells on the 1-cells $\mathcal{E}\mathbf{1}_n\{t\}$ and $\mathcal{F}\mathbf{1}_n\{t\}$ for every $t \in \mathbb{Z}$ as follows:

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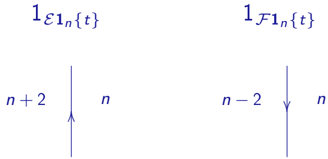

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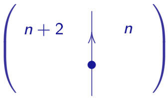
deg  $= 2,$

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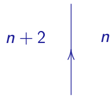

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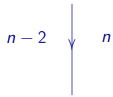

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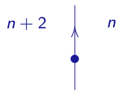
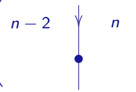
deg (

) = 2, similarly, deg (

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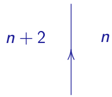

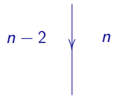

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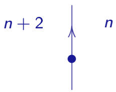
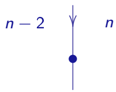
deg (

) = 2, similarly, deg (

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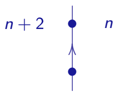
deg (

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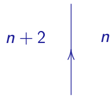

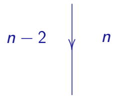

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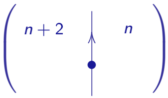
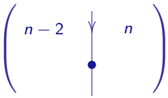
deg 

$= 2,$ similarly, deg 

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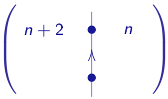
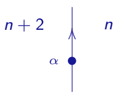
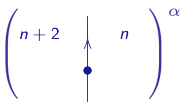
deg 

$= 4.$

  - For any $\alpha \in \mathbb{N}$, $\mathcal{U}(\mathfrak{sl}_2)(\mathcal{E}\mathbf{1}_n\{2\alpha\}, \mathcal{E}\mathbf{1}_n)$ is of dimension 1, generated by



$:=$


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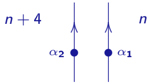
  - If $h(\alpha_1, \alpha_2) =$ 

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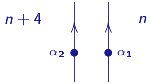
  - If $h(\alpha_1,\alpha_2) =$ 

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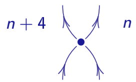
$$\deg\left(\quad\right) = -2.$$



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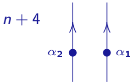
  - If $h(\alpha_1,\alpha_2) =$ 

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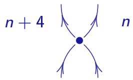
deg (

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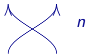
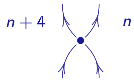


$:=$ 

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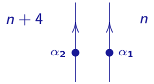
  - If $h(\alpha_1, \alpha_2) =$ 

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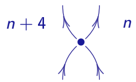
deg (

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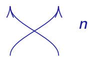
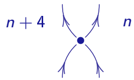
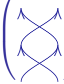
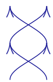


$:=$ 

  - One can deduce further relations:

deg () $= 4$ $\rightsquigarrow$  $= 0$

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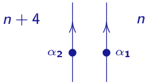
  - If $h(\alpha_1,\alpha_2) =$ 

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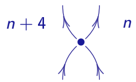
deg (

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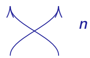
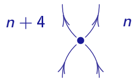


:= 

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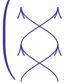
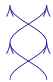
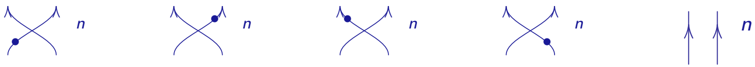
deg () $= 4$ $\rightsquigarrow$  $= 0$

  - Since the coeff before q^0 is 3, the diagrams



are not linearly independant. We add nil Hecke relations between these forms.

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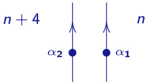
  - If $h(\alpha_1,\alpha_2) =$ 

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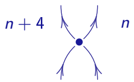
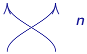
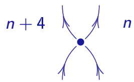
    deg (

) $= -2.$ 

    $:=$ 

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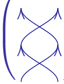
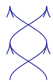
    deg () $= 4$ $\rightsquigarrow$  $= 0$
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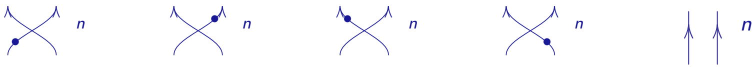
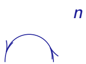
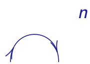
    

    are not linearly independant. We add nil Hecke relations between these forms.

- **Example 3:** $\langle FE1_n, 1_n \rangle = \frac{q^{1+n}}{1-q^2}$, $\langle EF1_n, 1_n \rangle = \frac{q^{1-n}}{1-q^2}$, $\langle 1_n, EF1_n \rangle = \frac{q^{1+n}}{1-q^2}$, $\langle 1_n, FE1_n \rangle = \frac{q^{1-n}}{1-q^2}$.

generator				
degree	1+n	1-n	1+n	1-n

subject to pivotal isotopy relations.

Categorification of $\dot{\mathrm{U}}$: 2-cells

- **Example 4:** $\langle EF1_n, FE1_n \rangle = (1+q^2)\left(\frac{1}{1-q^2}\right)^2$.

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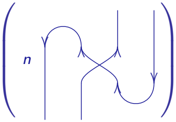
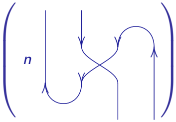
deg  = deg  $= 0$

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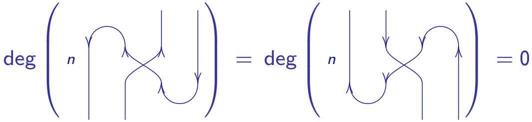

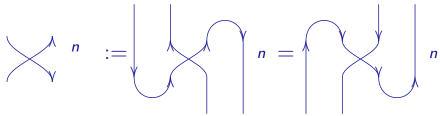

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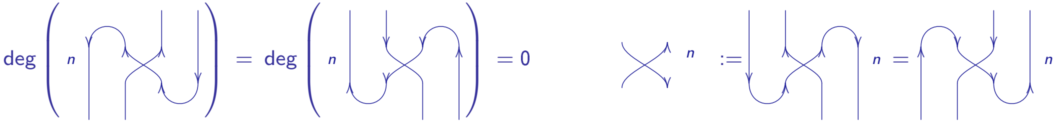

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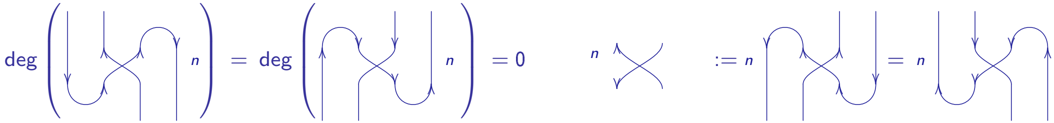

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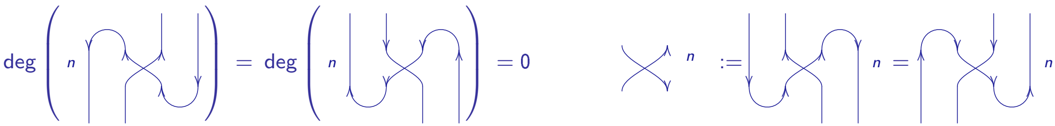

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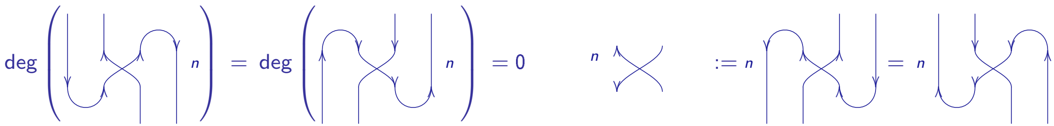
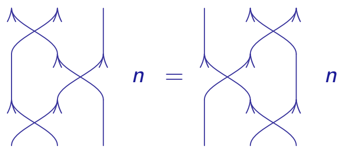


- One can show further relations and linear dependencies using Lusztig's pairing, *e.g.*



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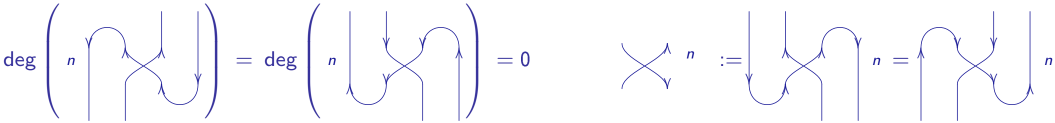

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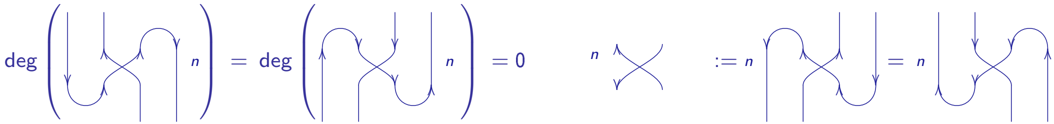

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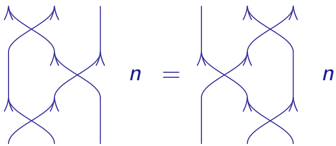
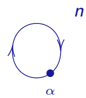
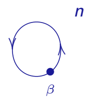


- It is hard to prove that one has obtained all the necessary generating 2-cells: Lauda proved that with the suited relations, the indecomposable 1-cells of $\mathcal{U}(\mathfrak{sl}_2)$ correspond up to shifts with elements of $\mathcal{B}$.

Bubbles

- A **bubble** is an element of the algebras $\mathrm{End}(\mathbf{1}_n)$ for $n \in \mathbb{Z}$:





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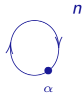
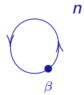
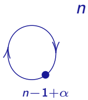
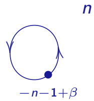




deg (

) $= 2\alpha$, deg (

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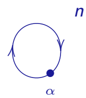
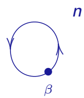
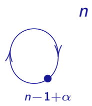
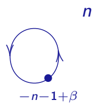




deg (

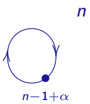
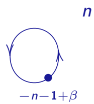
) $= 2\alpha$, deg (

) $= 2\beta$.

- There is a link between the bubble algebras $\mathrm{End}(\mathbf{1}_n)$ in $\mathcal{U}(\mathfrak{sl}_2)$ and the algebra of symmetric polynomials $\Lambda(x_1, \dots, x_n)$. This latter is generated by
  - elementary symmetric polynomials $e_r(x_1, \dots, x_n) = \sum_{j_1 < \cdots < j_r} x_{j_1} \dots x_{j_r}$
  - complete symmetric polynomials $h_r(x_1, \dots, x_n) = \sum_{m_1 + \cdots + m_n = r} x_1^{m_1} \dots x_n^{m_n}$

$$\sum_{k \geq 0} (-1)^k e_k h_{\alpha - k} = \delta_{\alpha,0} \text{ with } h_j = e_j = 0 \text{ for } j < 0 \text{ and } e_1 = h_1 = 1.$$

- A **bubble** is an element of the algebras $\mathrm{End}(\mathbf{1}_n)$ for $n \in \mathbb{Z}$:

deg () $= 2\alpha$, deg () $= 2\beta$.

- There is a link between the bubble algebras $\mathrm{End}(\mathbf{1}_n)$ in $\mathcal{U}(\mathfrak{sl}_2)$ and the algebra of symmetric polynomials $\Lambda(x_1, \ldots, x_n)$. This latter is generated by
  - elementary symmetric polynomials $e_r(x_1, \ldots, x_n) = \sum_{j_1 < \cdots < j_r} x_{j_1} \ldots x_{j_r}$
  - complete symmetric polynomials $h_r(x_1, \ldots, x_n) = \sum_{m_1 + \cdots + m_n = r} x_1^{m_1} \ldots x_n^{m_n}$

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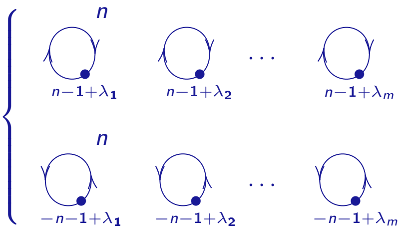
- For a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, define $e_\lambda := e_{\lambda_1} \ldots e_{\lambda_n}$, then there is an injective mapping

$$\phi^n : \Lambda(x_1, \ldots, x_n) \longrightarrow \mathrm{End}(\mathbf{1}_n)$$

$e_\lambda = e_{\lambda_1} \ldots e_{\lambda_m} \mapsto$



for $n > 0$

if $n < 0$.

and define $e_{\lambda,n} := \phi^n(e_\lambda)$.

Lifting of $\mathfrak{sl}_2$-relations

- To lift the relation $EF1_n - FE1_n = [n]1_n$ of $\dot{\mathbf{U}}$, one proves isomorphisms of the form

$$\mathcal{E}\mathcal{F}\mathbf{1}_n \cong \mathcal{F}\mathcal{E}\mathbf{1}_n \oplus \mathbf{1}_n^{\oplus[n]} \quad \text{for } n \geq 0, \quad \mathcal{F}\mathcal{E}\mathbf{1}_n \cong \mathcal{E}\mathcal{F}\mathbf{1}_n \oplus \mathbf{1}_n^{\oplus[-n]} \quad \text{for } n \leq 0.$$

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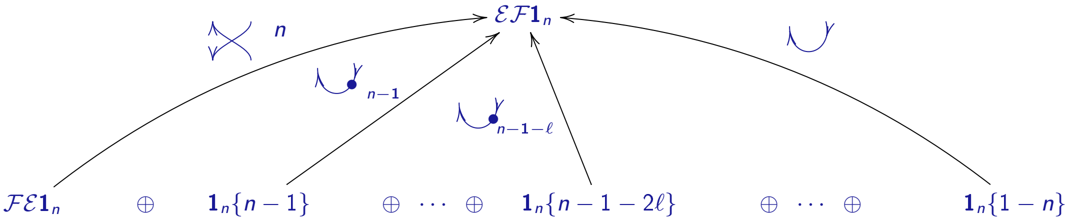

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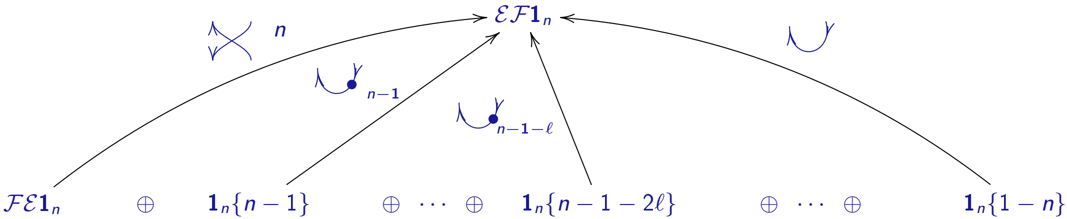

- We will explicit an inverse by its components on each summand: ζ_+^n for the $\mathcal{F}\mathcal{E}\mathbf{1}_n$ summand, and ζ_+^ℓ for $0 \leq l \leq n-1$ for other summands. Using Lusztig's pairing,

$$\zeta_+^\ell = \sum_{|\lambda|+j=\ell} \alpha_\lambda^\ell(n) e_{\lambda,n} \quad {}^{n}\; {}^{j}, \qquad \zeta_+^n = \quad n$$

for some coefficients $\alpha_i^\ell(n) \in \mathbb{K}$ that are determined by $\delta_{b,0} = \sum_{\lambda:|\lambda|\leq b} \alpha_\lambda^\ell(n) e_{\lambda,n} e_{b-|\lambda|,n}$.

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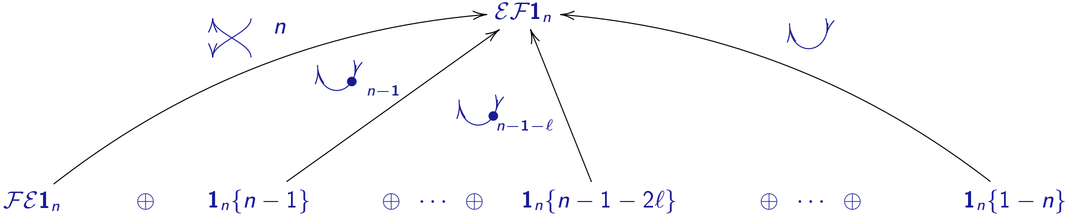

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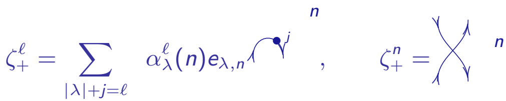

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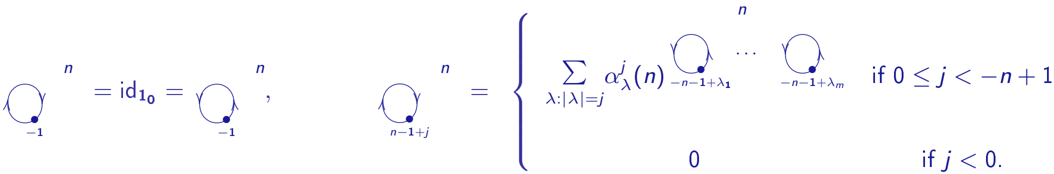
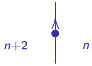
- Khovanov and Lauda introduced **fake bubbles** to obtain relations that are fully diagrammatic:



for $n = 0$ and $n < 0$ respectively, with a similar definition in the case $n > 0$.

Conclusion : generating 2-cells and relations

- $\mathcal{U}(\mathfrak{sl}_2)$ admits for generating 2-cells:



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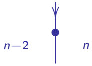

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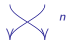

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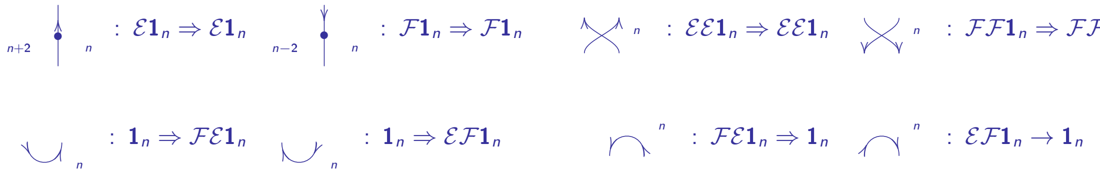
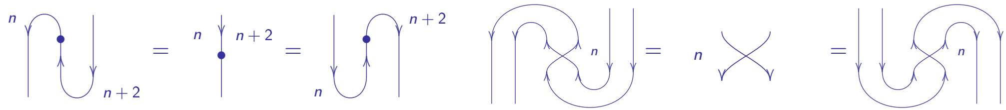
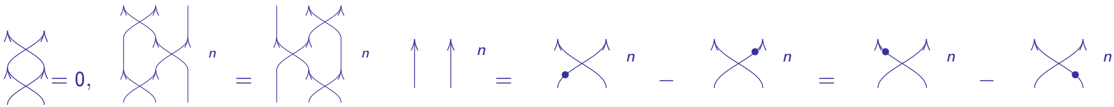


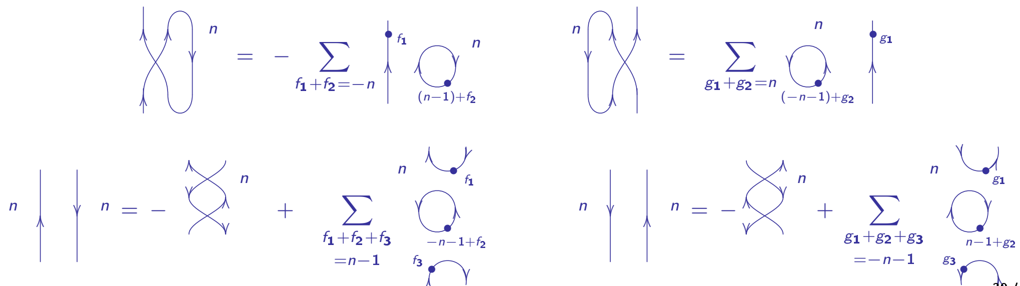
- These are subject to relations
  - isotopy relations for caps and cups, and cyclicity relations for dots and crossings:



  - nilHecke relations for diagrams with upward orientations:



  - Negative degree bubbles are 0, bubbles of degree 0 are identities, and infinite Grassmannian relation.
  - Quantum $\mathfrak{sl}_2$-relations:



for all $n \in \mathbb{Z}$. Whenever the summations are nonzero they utilize fake bubbles.

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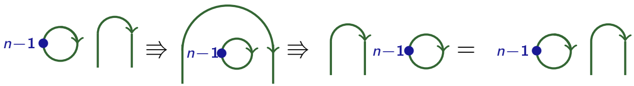

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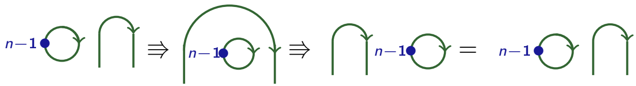

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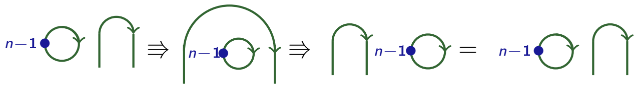

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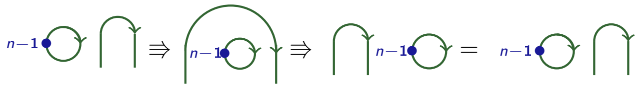

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  - However, can bring new shapes of rewriting cycles to take into account, and critical branchings are harder to list, since they consist in application of relations on two diagrams that are E-congruent.

Thank you for your attention.