3-polygraphs and Squier's completion Theorem

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Séminaire de réécriture algébrique

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I. 3-categories and 3-polygraphs

II. Coherence from convergence

III. Homology via Squier

IV. Coherent presentations of plactic monoids

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  - 2-cells = equivalence classes of 2-cells of $\mathcal{C}$ modulo $\equiv_\Gamma$

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  - *extended presentation* of categories

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- Thm (SOK): FDT is independent of choice of finite 2-polygraph.

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  that are *minimal*.
- Family of generating confluences of X is a cellular extension Γ of $X_2^\top$ containing precisely one 3-cell

$$\begin{tikzcd} & w' \arrow[dr, Rightarrow, dotted] & \\ w \arrow[ur, Rightarrow, "\alpha"] \arrow[dr, Rightarrow, "\beta"'] & & w''' \\ & w'' \arrow[ur, Rightarrow, dotted] & \end{tikzcd}$$

  for every element of $\mathrm{Crit}(X)$.

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- Question: How to constructively *choose* confluence diagrams for the critical branchings?

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- σ is called left-normalizing (resp. right-normalizing) if

$$\sigma_{uv} = (\sigma_u \star_0 v) \star_1 \sigma_{\widehat{u}v}, \quad (\text{resp.} \sigma_{uv} = (u \star_0 \sigma_v) \star_1 \sigma_{u\widehat{v}})$$

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- Rightmost normalization strategy

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- Denote by λ_u (resp. ρ_u) the minimal (resp. maximal) elements of this order.
- Leftmost normalization strategy

$$\sigma_u := \lambda_u \star_1 \sigma(t(\lambda_u))$$

- Rightmost normalization strategy

$$\sigma_u = \rho_u \star_1 \sigma_t(\rho_u)$$

- We can choose a family of generating confluences via the leftmost and rightmost normalization strategies.

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- Thus $(X, \{A\})$ with A the 3-cell above is a coherent presentation for M_3.
- **Remark**: Normalization strategies provide a way for specifying a family of generating confluences. The *shape* of such confluence diagrams depends on the intrinsic nature of the monoid and its combinatorics.

III. Homology via Squier

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- Given a free resolution of $\mathbb{Z}$, we associate the following chain complex

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- Homology of M with integral coefficients is defined by setting

$$H_n(M, \mathbb{Z}) = \ker(\widetilde{d}_n)/\mathrm{im}(\widetilde{d}_{n+1}),$$

  and we call it the n-th homology group of M.

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with the boundary maps given by

$$\varepsilon\left(\sum_{u\in M} a_u u\right) = \sum a_u, \quad d_1(uv) = [u] + \overline{u}[v], \; d_{i+1}(A) = [s_i(A)] - [t_i(A)]$$

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  with $i = 1, 2$
- If X is finite, then $\mathbb{Z}M[X_i]$ are finitely generated free $\mathbb{Z}M$-modules

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- consider the two directed labeled graphs

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- If $x \xrightarrow{i} y$, write $f_i.x = y$ and $e_i.y = x$.
- One defines a graph structure on A_n^* and C_n^* by defining

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$$e_i.(uv) = \begin{cases} (e_i.u)v & \text{if } \varphi_i(u) \geq \varepsilon_i(v), \\ u(e_i.v) & \text{if } \varphi_i(u) < \varepsilon_i(v), \end{cases}$$

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- Recall that these monoids admit finite convergent presentations, called the *column presentations*.

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- words of this form w^0 such that $e_i.w$ is undefined, are called *highest weights*

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A-trees and C-trees

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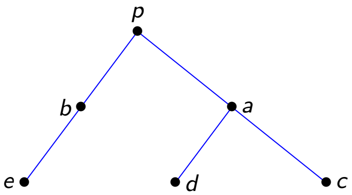
- The model for type A is of the form



and can be used to show that the confluence diagrams of critical branchings for $Cols(A_n)$ are of the form

$$\begin{tikzcd}
 & t'u'v \arrow[r, Rightarrow, "t'\alpha_{u'v}"] & t'u''v' \arrow[dr, Rightarrow, "\alpha_{t'u''v'}"] & \\
tuv \arrow[ur, Rightarrow, "\alpha_{tuv}"] \arrow[dr, Rightarrow, "t\alpha_{uv}"'] & & & t''u'''v' \\
 & tu_1v_1 \arrow[r, Rightarrow, "\alpha_{tu_1v_1}"] & t_1u_2v_2 \arrow[ur, Rightarrow, "t_1\alpha_{u_2v_2}"'] &
\end{tikzcd}$$

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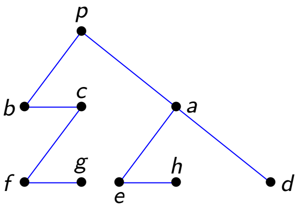
- The model for type C is of the form



and can be used to show that the confluence diagrams of critical branchings of $Cols(C_n)$ are of the form

$$\begin{tikzcd}
 & t'u'v \arrow[r, Rightarrow] & t'u''v' \arrow[r, Rightarrow] & t''u'''v' \arrow[dr, Rightarrow] & \\
tuv \arrow[ur, Rightarrow, "\alpha_{tu}v"] \arrow[dr, Rightarrow, "t\alpha_{uv}"'] & & & & t_0u_0v_0 \\
 & tu_1v_1 \arrow[rr, Rightarrow] & & t_1u_2v_1 \arrow[ur, Rightarrow] &
\end{tikzcd}$$

Thank you very much for your attention!