

# Computation of Koszul homology and application to partial differential systems

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Algebraic rewriting seminar

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## I. Motivations

- ▷ formal theory of PDEs systems
- ▷ formal integrability
- ▷ 2-acyclicity and Koszul homology

## II. Computation of Koszul homology

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- ▷ 2-acyclicity criterion

## III. Examples

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- ▷ parametric coefficients
- ▷ polynomial coefficients

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# I. MOTIVATIONS

Consider a PDEs system

$$(\Sigma) : F^p \left( x^i, \frac{\partial^{|\mu|} u^j}{\partial x^\mu} \right) = 0$$

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→  $F^1, \dots, F^q$ : equations of the system

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Geometric methods

- integral manifolds
- differential forms
- jet bundles

Algebraic methods

- differential elimination
- integrability conditions
- homological algebra

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Nonlinear control problems

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### Examples

$$(\Sigma_1): \dot{u} = u$$

→  $(\Sigma_1)$  is formally integrable with formal power series solutions

$$u_0 + u_0 t + \frac{u_0}{2} t^2 + \frac{u_0}{3!} t^3 + \dots$$

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$$(\Sigma_2): u_{33} = u_{13}, \quad u_{23} = u_{13}, \quad u_{12} = u_{11}$$

→ derivatives of  $(\Sigma_2)$  yield coherent results, e.g.,

$$(u_{33})_2 = u_{123} = u_{113} \quad \text{and} \quad (u_{23})_3 = u_{133} = u_{113}$$

→  $(\Sigma_2)$  is formally integrable with formal power series solutions

$$u_0 + u_i x^i + \frac{u_{11}}{2} (x^1)^2 + u_{11} x^1 x^2 + u_{13} x^1 x^3 + \frac{u_{22}}{2} (x^2)^2 + u_{13} x^2 x^3 + \frac{u_{13}}{2} (x^3)^2 + \dots$$

### Integrability defect and homology

**Janet example.**  $(\Sigma) : u_{33} = x^2 u_{11}, \quad u_{22} = 0$

→  $(u_{22})_{33} = 0$  and

$$(u_{33})_{22} = \frac{\partial}{\partial x^2} (x^2 u_{112} + u_{11}) = x^2 u_{1122} + 2u_{112} = 2u_{112}$$

→ new integrability condition  $u_{112} = 0$

**Remark.** The Spencer cohomology does not vanish in homological degree 2

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effectively check 2-acyclicity of linear systems

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## II. COMPUTATION OF KOSZUL HOMOLOGY

Let a system  $(\Sigma) : \forall 1 \leq i \leq q, \quad R_1^i u^1 + \cdots + R_p^i u^p = 0$

→ described by  $(R_j^i) \in \mathcal{D}^{q \times p}$  ( $\mathcal{D}$ : ring of partial differential operators)

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**Symbol module:**  $\mathcal{M}(\Sigma) := \mathcal{A}^{1 \times p} / (\mathcal{A}^{1 \times q} \sigma(R))$

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**Koszul homology:** homology  $H_k \simeq \ker(\partial_k) / \text{im}(\partial_{k+1})$  of the **Koszul complex**

$$0 \rightarrow \Lambda^n T \otimes \mathcal{M}(\Sigma) \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_3} \Lambda^2 T \otimes \mathcal{M}(\Sigma) \xrightarrow{\partial_2} T \otimes \mathcal{M}(\Sigma) \xrightarrow{\partial_1} \mathcal{M}(\Sigma) \rightarrow 0$$

where for basis elements  $(\chi_i)$  of  $T = \mathcal{A}_1^n$

$$\partial_k (\chi_1 \wedge \cdots \wedge \chi_k \otimes m) := \sum_{i=1}^k (-1)^{i+1} \chi_1 \wedge \cdots \wedge \widehat{\chi}_i \wedge \cdots \wedge \chi_k \otimes \chi_i m$$



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**Definitions.**  $\mathcal{M}(\Sigma)$  is **2-acyclic** if  $H_2$  vanishes, and **involutiv** if  $H_2, \dots, H_n$  vanish

**Objective:** compute  $H_2 \rightarrow$  using OreMorphisms package

**Approach:** given  $\mathcal{M}_2 \xrightarrow{f} \mathcal{M}_1 \xrightarrow{g} \mathcal{M}_0$  s.t.  $g \circ f = 0$  and  $\mathcal{M}_i = \mathcal{A}^{1 \times p_i} / (\mathcal{A}^{1 \times q_i} R_i)$

$$\begin{array}{ccccccc}
 \mathcal{A}^{1 \times q_2} & \xrightarrow{\cdot R_2} & \mathcal{A}^{1 \times p_2} & \longrightarrow & \mathcal{M}_2 & \longrightarrow & 0 \\
 \cdot Q_2 \downarrow & & \cdot P_2 \downarrow & & f \downarrow & & \\
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 \end{array}$$

OreMorphisms computes matrices  $S', S''$  s.t.  $\ker \begin{pmatrix} \cdot P_1 \\ \cdot R_0 \end{pmatrix} = \mathcal{A}^{1 \times s'} (S' \quad -S'')$  and

$$\ker(g)/\text{im}(f) = \mathcal{A}^{1 \times s'} S' / \left( \mathcal{A}^{1 \times (p_2 + q_2)} \begin{pmatrix} P_2 \\ R_1 \end{pmatrix} \right)$$

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**Case of  $H_2$ :**  $\mathcal{M}_i = \Lambda^i T \otimes \mathcal{M}(\Sigma)$ ,  $f = \partial_{i+1}$ ,  $g = \partial_i$

$\rightarrow R_i = I \otimes \sigma(R)$ ;  $P_i, Q_i = (\text{grad}, \text{curl}, \text{div}) \otimes I$

**Theorem (C., Cluzeau, Quadrat).** Let  $(\Sigma)$  be a linear system of PDEs and let  $\mathcal{M}(\Sigma)$  be the symbol module of  $(\Sigma)$ . Let  $S, S'$  be two matrices such that

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**Algorithmic side of the theorem**

Existence of a right factorization  
may be checked using OreModules

### III. Examples

$(\Sigma) : u_{33} = u_{13}, u_{23} = u_{13}, u_{12} = u_{11}$

**Symbol module:**  $\mathcal{A} := \mathbb{Q}[\chi_1, \chi_2, \chi_3]$

$$\mathcal{M}(\Sigma) = \mathcal{A} / \left( \mathcal{A}^{1 \times 3} \begin{pmatrix} \chi_3^2 - \chi_1 \chi_3 \\ \chi_2 \chi_3 - \chi_1 \chi_3 \\ \chi_1 \chi_2 - \chi_1^2 \end{pmatrix} \right)$$

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$$\text{2nd homology group: } H_2 = \mathcal{A}^{1 \times 7} S' / (\mathcal{A}^{1 \times 10} S)$$

$$S' = \begin{pmatrix} \chi_3 & \chi_3 & \chi_3 \\ \chi_1 & \chi_2 & \chi_3 \\ 0 & \chi_1 - \chi_2 & 0 \\ 0 & \chi_2 \chi_3 - \chi_3^2 & 0 \\ 0 & 0 & \chi_2 \chi_3 - \chi_3^2 \\ 0 & 0 & \chi_1 \chi_3 - \chi_3^2 \\ 0 & 0 & \chi_1^2 - \chi_1 \chi_2 \end{pmatrix} \quad S = \begin{pmatrix} \chi_1 & \chi_2 & \chi_3 \\ \chi_3^2 - \chi_1 \chi_3 & 0 & 0 \\ \chi_2 \chi_3 - \chi_1 \chi_3 & 0 & 0 \\ \chi_1 \chi_2 - \chi_1^2 & 0 & 0 \\ 0 & \chi_3^2 - \chi_1 \chi_3 & 0 \\ 0 & \chi_2 \chi_3 - \chi_1 \chi_3 & 0 \\ 0 & \chi_1 \chi_2 - \chi_1^2 & 0 \\ 0 & 0 & \chi_3^2 - \chi_1 \chi_3 \\ 0 & 0 & \chi_2 \chi_3 - \chi_1 \chi_3 \\ 0 & 0 & \chi_1 \chi_2 - \chi_1^2 \end{pmatrix}$$



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2-acyclicity test: **success**

$$(\Sigma) : u_{33} = u_{22} = u_{13} = u_{12} = 0, u_{23} = \alpha u_{11} \quad (\alpha: \text{a parameter})$$

**Symbol module:**  $\mathcal{A} := \mathbb{Q}[\alpha][\chi_1, \chi_2, \chi_3]$

$$\mathcal{M}_\alpha(\Sigma) = \mathcal{A} / \left( \mathcal{A}^{1 \times 5} \begin{pmatrix} \chi_3^2 & \chi_2 \chi_3 - \alpha \chi_1^2 & \chi_2^2 & \chi_1 \chi_3 & \chi_1 \chi_2 \end{pmatrix}^T \right)$$

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2nd homology group:  $H_2 = \mathcal{A}^{1 \times 11} S' / (\mathcal{A}^{1 \times 16} S)$

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2nd homology group:  $H_2 = \mathcal{A}^{1 \times 11} S' / (\mathcal{A}^{1 \times 16} S)$

$$S' = \begin{pmatrix} x_2 & 0 & \alpha x_1 \\ x_3 & \alpha x_1 & 0 \\ x_1 & 0 & 0 \\ x_3^2 & 0 & 0 \\ x_2 x_3 & 0 & 0 \\ x_2^2 & 0 & 0 \\ 0 & x_3 & 0 \\ 0 & x_2 & x_3 \\ 0 & 0 & x_2^2 \\ 0 & 0 & x_3^2 \\ 0 & 0 & x_1 x_3 \end{pmatrix} \quad S = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_3^2 & 0 & 0 \\ x_2 x_3 - \alpha x_1^2 & 0 & 0 \\ x_2^2 & 0 & 0 \\ x_1 x_3 & 0 & 0 \\ x_1 x_2 & 0 & 0 \\ 0 & x_3^2 & 0 \\ 0 & x_2 x_3 - \alpha x_1^2 & 0 \\ 0 & x_2^2 & 0 \\ 0 & x_1 x_3 & 0 \\ 0 & x_1 x_2 & 0 \\ 0 & 0 & x_3^2 \\ 0 & 0 & x_2 x_3 - \alpha x_1^2 \\ 0 & 0 & x_2^2 \\ 0 & 0 & x_1 x_3 \\ 0 & 0 & x_1 x_2 \end{pmatrix}$$

2-acyclicity test: **success**. Moreover,  $\mathcal{M}_\alpha(\Sigma)$  is **involution** iff  $\alpha \neq 0$

$$(\Sigma) : u_{33} = x^2 u_{11}, u_{22} = 0$$

**Symbol module:**  $\mathcal{A} := \mathbb{Q}[x^1, x^2, x^3][\chi_1, \chi_2, \chi_3]$

$$\mathcal{M}_\alpha(\Sigma) = \mathcal{A} / \left( \mathcal{A}^{1 \times 2} \begin{pmatrix} \chi_3^2 - x^2 \chi_1^2 \\ \chi_2^2 \end{pmatrix} \right)$$

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**2-acyclicity test:** fail



## **IV. CONCLUSION AND PERSPECTIVES**

## Conclusion and perspectives

### Summary of presented results

- new criterion for 2-acyclicity of linear PDEs systems
- effective test using `OreMorphisms` and `OreModules` packages
- illustration with various classes of linear systems

**Remark.** More generally, involutivity can be checked with `OreMorphism` and `OreModules`

### Further works

- go further in the effective approach to Spencer cohomology  
(Koszul-Tate theory, Spencer sequences)
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(elasticity theory, hydrodynamics, electromagnetism, general relativity)

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**THANK YOU FOR LISTENING!**