Central Submonads and Notions of Computation

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April, 27th 2023

Overview

- For any monoid M, its centre $Z(M)$ is a commutative submonoid;
- For any semiring R, its centre $Z(R)$ is a commutative subsemiring.
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Context:

- a symmetric monoidal category $(\mathbf{C}, I, \otimes)$,
- a strong monad $(\mathcal{T}, \eta, \mu, \tau)$.

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We wonder:

- Is there a commutative submonad of $\mathcal{T}$ which is its centre? When does it exist?
- Is there an appropriate computational interpretation?

Background

Safe Net: functors, monoidal category

Category $\mathbf{C}$: some objects $Obj(\mathbf{C})$, and morphisms between them. Given two objects A, B; $\mathbf{C}(A, B)$ is the *set* of morphisms $A \to B$.

Morphisms can be composed: $f : A \to B$ and $g : B \to C$ give rise to $g \circ f : A \to C$.

A functor $F : \mathbf{C} \to \mathbf{C}$ is a function on objects <u>and</u> on morphisms. Given $f : A \to B$, $F(f)$ is a morphism $F(A) \to F(B)$. Functors preserve composition: $F(g \circ f) = F(g) \circ F(f)$.

We will write $Ff : FA \to FB$.

Monoidal structure: seen as a *tensor* product.

Objects can be tensored: $A \otimes B$.

Morphisms can be tensored $f \otimes g$.

$\otimes$ is a (bi)functor.

An easy way out?

A monad is a monoid in the category of endofunctors, wrt composition.

But it is not a monoid as a math undergrade would understand it.

It is a monoid *object* M in $\mathbf{C}^{\mathbf{C}}$, with a unit $\eta : I \to M$ and a multiplication $\mu : M \circ M \to M$. (with coherence conditions)

What is the centre of a monoid-object?

The Strength of a Monad

- Given a monoid M, its centre is defined as

$$Z(M) \stackrel{\text{def}}{=} \{x \in M \mid \forall y \in M.\ x \cdot y = y \cdot x\}.$$

- Notice there is an implicit *swap* in the arguments.
- *But*, the definition of a monad is independent of any monoidal structure on the base category.
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- Unclear how to define a suitable notion of centre for such monads.
- Instead, we introduce the centre for *strong* monads acting on symmetric monoidal categories.
- The monadic strength is a natural transformation $\tau_{X,Y} \colon X \otimes \mathcal{T}Y \to \mathcal{T}(X \otimes Y)$ that satisfies some coherence conditions w.r.t. monoidal structure.
- The monadic left strength is a natural transformation $\tau'_{X,Y} \colon \mathcal{T}X \otimes Y \to \mathcal{T}(X \otimes Y)$ that may be defined via τ and the monoidal symmetry.

Commutative Monads

Definition (Commutative Monad)

A strong monad $\mathcal{T}$ is said to be *commutative* if the following diagram:

$$
\begin{array}{ccccc}
\mathcal{T}X \otimes \mathcal{T}Y & \xrightarrow{\tau_{\mathcal{T}X,Y}} & \mathcal{T}(\mathcal{T}X \otimes Y) & \xrightarrow{\mathcal{T}\tau'_{X,Y}} & \mathcal{T}^2(X \otimes Y) \\
\big\downarrow{\scriptstyle \tau'_{X,\mathcal{T}Y}} & & & & \big\downarrow{\scriptstyle \mu_{X\otimes Y}} \\
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\end{array}
$$

commutes for every choice of objects X and Y.

The Centre of a Monad on Set

The first example

Given a monoid (M, e, m), the writer monad: $(M \times -) : \mathbf{Set} \to \mathbf{Set}$ has the following monad structure:

- $\eta_X : X \to M \times X :: x \mapsto (e, x)$;
- $\mu_X : M \times (M \times X) \to M \times X :: (z, (z', x)) \mapsto (m(z, z'), x)$,
- $\tau_{X,Y} : X \times (M \times Y) \to M \times (X \times Y) :: (x, (z, y)) \mapsto (z, (x, y))$.

What should be the centre? What about $Z(M) \times -$?
Indeed, it is a commutative submonad of $(M \times -)$.

Commutative Monads in Set

$\mathcal{T}$: **Set** $\to$ **Set** is said to be *commutative* if the following diagram:

$$
\begin{array}{ccccc}
\mathcal{T}X \times \mathcal{T}Y & \xrightarrow{\tau_{\mathcal{T}X,Y}} & \mathcal{T}(\mathcal{T}X \times Y) & \xrightarrow{\mathcal{T}\tau'_{X,Y}} & \mathcal{T}^2(X \times Y) \\
{\scriptstyle \tau'_{X,\mathcal{T}Y}} \downarrow & & & & \downarrow {\scriptstyle \mu_{X \times Y}} \\
\mathcal{T}(X \times \mathcal{T}Y) & \xrightarrow[\mathcal{T}\tau_{X,Y}]{} & \mathcal{T}^2(X \times Y) & \xrightarrow[\mu_{X \times Y}]{} & \mathcal{T}(X \times Y)
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commutes for every choice of sets X and Y.
How would you define the centre $\mathcal{Z}$ of $\mathcal{T}$?

Central Subset

The trick is to consider all the monadic elements of $\mathcal{T}X$ that make the previous diagram commute.

Definition (Centre)

Given a set X, the *centre* of $\mathcal{T}$ at X, written $\mathcal{Z}X$, is defined to be the set

$$\mathcal{Z}X \stackrel{\text{def}}{=} \{t \in \mathcal{T}X \mid \forall Y \in \mathrm{Ob}(\mathbf{Set}).\forall s \in \mathcal{T}Y.$$
$$\mu(\mathcal{T}\tau'(\tau(t, s))) = \mu(\mathcal{T}\tau(\tau'(t, s)))\}.$$

We write $\iota_X : \mathcal{Z}X \subseteq \mathcal{T}X$ for the indicated subset inclusion.

The Centre

- **Lemma:** The assignment $\mathcal{Z}(-)$ extends to a functor $\mathcal{Z} : \mathbf{Set} \to \mathbf{Set}$ when we define
$$\mathcal{Z}f \stackrel{\text{def}}{=} \mathcal{T}f|_{\mathcal{Z}X} : \mathcal{Z}X \to \mathcal{Z}Y,$$
for any function $f : X \to Y$.
- **Lemma:** For any two sets X and Y, the monadic unit $\eta_X : X \to \mathcal{T}X$, the monadic multiplication $\mu_X : \mathcal{T}^2 X \to \mathcal{T}X$, and the monadic strength $\tau_{X,Y} : X \times \mathcal{T}Y \to \mathcal{T}(X \times Y)$ (co)restrict respectively to functions $\eta^{\mathcal{Z}}_X : X \to \mathcal{Z}X$, $\mu^{\mathcal{Z}}_X : \mathcal{Z}^2 X \to \mathcal{Z}X$ and $\tau^{\mathcal{Z}}_{X,Y} : X \times \mathcal{Z}Y \to \mathcal{Z}(X \times Y)$.
- **Theorem:** The assignment $\mathcal{Z}(-)$ extends to a *commutative submonad* $(\mathcal{Z}, \eta^{\mathcal{Z}}, \mu^{\mathcal{Z}}, \tau^{\mathcal{Z}})$ of $\mathcal{T}$ with $\iota_X : \mathcal{Z}X \subseteq \mathcal{T}X$ the submonad morphism. Furthermore, there exists a canonical[1] isomorphism $\mathbf{Set}_{\mathcal{Z}} \cong Z(\mathbf{Set}_{\mathcal{T}})$.

[1] Details later.

Examples

- Continuation monad: $\mathcal{T} = [[-, S], S] : \mathbf{Set} \to \mathbf{Set}$.

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  - The centre is naturally isomorphic to the *identity monad*; therefore the centre is trivial.
- If $\mathcal{T}$ is commutative, its centre is itself.
- The centre of $(M \times -)$ is indeed $(Z(M) \times -)$.

Link with Lawvere theories

- In a Lawvere theory $\mathbf{T}$, we say that $f\colon A^n \to A^{n'}$ and $g : A^m \to A^{m'}$ commute if and only if $f^{m'} \circ g^n$ (also written $f \star g$) and $g^{n'} \circ f^m$ (also written $g \star f$) are equal, up to isomorphism.
- If $\mathbf{S}$ is a subcategory of $\mathbf{T}$, the commutant of $\mathbf{S}$ in $\mathbf{T}$ is a subcategory of $\mathbf{T}$ whose morphisms commute with the morphisms of $\mathbf{S}$. This commutant is written $\mathbf{S}^\perp$, and is also a Lawvere subtheory of $\mathbf{T}$.
- Considering this, $\mathbf{T}^\perp$ is seen as the *centre* of the Lawvere theory $\mathbf{T}$.
- From $\mathbf{T}$ arises a finitery strong monad $\mathcal{T}$ on **Set**, and its centre $\mathcal{Z}$ is the monad of $\mathbf{T}^\perp$.

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Probably something to say about operads (with promonads?), but I have not figured this out yet.

Central Submonads in Symmetric Monoidal Categories

Central cones

Definition (Central Cone)

A *central cone* of $\mathcal{T}$ at X is given by a pair (Z, ι), an object Z and a morphism $\iota : Z \to \mathcal{T}X$, such that the diagram:

$$
\begin{array}{ccccc}
Z \otimes \mathcal{T}Y & \xrightarrow{\iota \otimes \mathcal{T}Y} & \mathcal{T}X \otimes \mathcal{T}Y & \xrightarrow{\tau'_{X,\mathcal{T}Y}} & \mathcal{T}(X \otimes \mathcal{T}Y) \\
\downarrow{\scriptstyle \iota \otimes \mathcal{T}Y} & & & & \downarrow{\scriptstyle \mathcal{T}\tau_{X,Y}} \\
\mathcal{T}X \otimes \mathcal{T}Y & & & & \mathcal{T}^2(X \otimes Y) \\
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commutes.

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Definition (Central Submonad)

Given a strong monad $(\mathcal{S}, \eta^{\mathcal{S}}, \mu^{\mathcal{S}}, \tau^{\mathcal{S}})$ which is a submonad of $\mathcal{T}$ with monad monomorphism ι, we say that $\mathcal{S}$ is a central submonad of $\mathcal{T}$ if for any object X, $(\mathcal{S}X, \iota_X)$ is a central cone for $\mathcal{T}$ at X. Besides, this last condition implies that $\mathcal{S}$ is commutative.

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- At first we thought that: there always is at least one central submonad for $\mathcal{T}$, but it is wrong.
- They form a category with strong monad morphisms. If the category has a terminal object, the latter is the centre of $\mathcal{T}$.

Centralisable Monads in Symmetric Monoidal Categories

Centralisable Monad

If (Z, ι) and (Z', ι') are two central cones of $\mathcal{T}$ at X, then a *morphism of central cones* $\varphi : (Z', \iota') \to (Z, \iota)$ is a morphism $\varphi : Z' \to Z$, such that $\iota \circ \varphi = \iota'$.

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Definition

We say that the monad $\mathcal{T}$ is *centralisable* if for any object X, a terminal central cone of $\mathcal{T}$ at X exists. We write $(\mathcal{Z}X, \iota_X)$ for the terminal central cone of $\mathcal{T}$ at X.

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The assignment $\mathcal{Z}(-)$ extends to a commutative submonad $(\mathcal{Z}, \eta^{\mathcal{Z}}, \mu^{\mathcal{Z}}, \tau^{\mathcal{Z}})$ of $\mathcal{T}$ with $\iota : \mathcal{Z} \Rightarrow \mathcal{T}$ the submonad monomorphism.

The Kleisli category $\mathbf{C}_{\mathcal{T}}$ of a monad $\mathcal{T}$ has the same objects as $\mathbf{C}$ and morphisms $A \to_{\mathcal{T}} B$ are the ones $A \to \mathcal{T}B$. Note that a submonad morphism induces a canonical embedding $\mathcal{I} : \mathbf{C}_{\mathcal{Z}} \to \mathbf{C}_{\mathcal{T}}$.

Kleisli Categories and Premonoidal Categories

Premonoidal category

- If $\mathbf{C}$ is symmetric monoidal and $\mathcal{T} : \mathbf{C} \to \mathbf{C}$ a strong monad;
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- there are two families of functors $(- \otimes_l X')$ and $(X \otimes_r -)$ on $\mathbf{C}_{\mathcal{T}}$.

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Definition (Central morphism [Power and Robinson, 1997])

A morphism $f \colon X \to Y$ in $\mathbf{C}_{\mathcal{T}}$ is *central* if for any morphism $f' : X' \to Y'$

in $\mathbf{C}_{\mathcal{T}}$, the following diagram:

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Central cones and central morphisms are actually equivalent notions!

Premonoidal Centre

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Proposition

If the strong monad $\mathcal{T}$ is centralisable, then the canonical embedding $\mathcal{I} : \mathbf{C}_{\mathcal{Z}} \to \mathbf{C}_{\mathcal{T}}$ corestricts to an isomorphism of categories $\hat{\mathcal{I}} : \mathbf{C}_{\mathcal{Z}} \to Z(\mathbf{C}_{\mathcal{T}})$.

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This is why we call $\mathcal{Z}$ <u>the</u> centre of $\mathcal{T}$.

Premonoidal adjunction

Kleisli adjunction

- In the Kleisli adjunction between $\mathbf{C}$ and $\mathbf{C}_{\mathcal{T}}$, the left adjoint, $\mathcal{J} : \mathbf{C} \to \mathbf{C}_{\mathcal{T}}$ always corestricts to $\hat{\mathcal{J}} : \mathbf{C} \to Z(\mathbf{C}_{\mathcal{T}})$.

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Proposition

If the strong monad $\mathcal{T}$ is centralisable, then $\hat{\mathcal{J}}$ is also a left adjoint and the adjunction induces the centre $\mathcal{Z}$.

Characterisation

The Main Theorem

Theorem (Centralisability)

Let $\mathbf{C}$ *be a symmetric monoidal category and* $\mathcal{T}$ *a strong monad on it. The following are equivalent:*

1. *For any object* X *of* $\mathbf{C}$, $\mathcal{T}$ *admits a terminal central cone at* X*;*
2. *There exists a commutative submonad* $\mathcal{Z}$ *of* $\mathcal{T}$ *such that the canonical embedding functor* $\mathcal{I} : \mathbf{C}_{\mathcal{Z}} \to \mathbf{C}_{\mathcal{T}}$ *corestricts to an isomorphism of categories* $\mathbf{C}_{\mathcal{Z}} \cong Z(\mathbf{C}_{\mathcal{T}})$*;*
3. *The corestriction of the Kleisli left adjoint* $\mathcal{J} : \mathbf{C} \to \mathbf{C}_{\mathcal{T}}$ *to the premonoidal centre* $\hat{\mathcal{J}} : \mathbf{C} \to Z(\mathbf{C}_{\mathcal{T}})$ *also is a left adjoint.*

Some Centralisable Monads and a non Centralisable one

- Using the main theorem, it follows every strong monad on many categories of interest (e.g., **Set**, **DCPO**, **Meas**, **Top**, **Hilb**, **Vect**) is centralisable.
- If **C** is a symmetric monoidal closed category that is total, then every strong monad on it is centralisable.
- If $\mathcal{T}$ is a commutative monad, then $\mathcal{T}$ is centralisable and its centre coincides with itself.

Is every strong monad centralisable?

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Is every strong monad centralisable? No!
Example built with a full subcategory **C** of **Set** where not all subsets of $\mathcal{T}X$ are objects of **C**.

More monads with non-trivial centres

Example

The valuation monad $\mathcal{V}$: **DCPO** $\to$ **DCPO** is strong, but its commutativity is an open problem [Jones, 1990]. The central submonad of $\mathcal{V}$ is precisely the "central valuations monad" described in [Jia et al., 2021].

Computational interpretation

A meta language

Type theory: a term judgement is given by: a context Γ of variables in the term, a term M define by a grammar (soon below), a type. A judgement is written: $\Gamma \vdash M : A$.

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\text{(Terms)} & M, N & ::= x \mid * \mid \lambda x^A.M \mid MN \mid \langle M, N \rangle \mid \pi_i M
\end{array}
$$

$$
\frac{}{\Gamma, x : A \vdash x : A}
$$

$$
\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x^A.M : A \to B} \qquad \frac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B}
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\text{(Terms)} & M, N & ::= x \mid * \mid \lambda x^A.M \mid MN \mid \langle M, N \rangle \mid \pi_i M \\
& & \mid \mathtt{ret}_{\mathcal{S}}\ M \mid \iota M \mid \mathtt{ret}_{\mathcal{T}}\ M \mid \mathtt{do}_{\mathcal{S}}\ x \leftarrow M;\ N \mid \mathtt{do}_{\mathcal{T}}\ x \leftarrow M;\ N
\end{array}
$$

$$\frac{}{\Gamma, x : A \vdash x : A}$$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x^A.M : A \to B} \qquad \frac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B}$$

$$\frac{\Gamma \vdash M : A}{\Gamma \vdash \mathtt{ret}_{\mathcal{S}}\ M : \mathcal{S}A} \qquad \frac{\Gamma \vdash M : \mathcal{S}A \quad \Gamma, x : A \vdash N : \mathcal{S}B}{\Gamma \vdash \mathtt{do}_{\mathcal{S}}\ x \leftarrow M;\ N : \mathcal{S}B}$$

$$\frac{\Gamma \vdash M : \mathcal{S}A}{\Gamma \vdash \iota M : \mathcal{T}A} \qquad \frac{\Gamma \vdash M : \mathcal{T}A \quad \Gamma, x : A \vdash N : \mathcal{T}B}{\Gamma \vdash \mathtt{do}_{\mathcal{T}}\ x \leftarrow M;\ N : \mathcal{T}B}$$

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Two types of semantics:

- One is syntactic: equational theory,
- The other is categorical: denotational semantics.

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$$\frac{\Gamma \vdash M : \mathcal{S}A \quad \Gamma \vdash N : \mathcal{T}B \quad \Gamma, x : A, y : B \vdash P : \mathcal{T}C}{\Gamma \vdash \mathtt{do}_{\mathcal{T}}\ x \leftarrow \iota M;\ \mathtt{do}_{\mathcal{T}}\ y \leftarrow N;\ P = \mathtt{do}_{\mathcal{T}}\ y \leftarrow N;\ \mathtt{do}_{\mathcal{T}}\ x \leftarrow \iota M;\ P : \mathcal{T}C}\ (\mathit{centr})$$

A theory is this calculus with additional *constants* and equational rules.

Denotational semantics

We write an interpretation $[\![-]\!]$.

Types A are interpretated as objects $[\![A]\!]$ in a category **C**.

Term judgements $\Gamma \vdash M : A$ as morphisms $[\![\Gamma \vdash M : A]\!] : [\![\Gamma]\!] \to [\![A]\!]$.

An interpretation of a theory is *sound* if:

$$\Gamma \vdash M = N : A \text{ implies } [\![\Gamma \vdash M : A]\!] = [\![\Gamma \vdash N : A]\!].$$

An interpretation is complete if:

$$\Gamma \vdash M = N : A \text{ iff } [\![\Gamma \vdash M : A]\!] = [\![\Gamma \vdash N : A]\!].$$

If the interpretation of a theory is sound and complete, it is safe to call **C** a *model* of the theory. The types $\mathcal{S}A$ and $\mathcal{T}A$ give rise to monads in **C** and better: $\mathcal{S}$ is a central submonad of $\mathcal{T}$!!

Equivalence

Theories form a 2-category **Th**. Models form a 2-category **Mod**.

and...

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Theorem

Th *and* **Mod** *are 2-equivalent.*

Computational use case for the centre of a monad

```
do                          do
  x <- op1                    y <- op2
  y <- op2                    x <- op1
  f x y                       f x y
```

If *at least one* of op1 or op2 is central, then the two programs are contextually equivalent!

Ongoing and Future Work

- Notion of Commutant for (pro)monads in general;
- Link with Garner's results on commutativity [Garner and Franco, 2016].

Thank you!

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