2-POLYGRAPHS AND STRING REWRITING

Illustration with plactic monoids

Nohra Hage

Université Saint-Joseph de Beyrouth (USJ)
École supérieure d'ingénieurs de Beyrouth (ESIB)

January 21, 2020
Beirut

Séminaire de réécriture algébrique

Plan :

1. **Two dimensional categories and polygraphs**

2. **Rewriting properties of 2 polygraphs**

   - **The Knuth–Bendix's completion**

3. **Column presentation for the plactic monoid of type A**

4. **Coherence**

Plan :

1. **Two dimensional categories and polygraphs**

Two dimensional categories and polygraphs

- A **1-category** is a data $\mathbf{C}$ made of :
  - a set $\mathbf{C}_0$ of **0-cells** of $\mathbf{C}$,
  - for every 0-cells x and y, a set $\mathbf{C}(x, y)$ of **1-cells** from x to y.
  - for every 0-cells x, y and z, a **0-composition** map

$$\star_0 : \mathbf{C}(x, y) \times \mathbf{C}(y, z) \longrightarrow \mathbf{C}(x, z),$$

  - for every 0-cell x, a specified element 1_x of $\mathbf{C}(x, x)$,
  - such that
    - the composition is **associative** :

$$((u \star_0 v) \star_0 w) \;=\; (u \star_0 (v \star_0 w)),$$

    for every 0-cells x, y, z, t, and 1-cells $u : x \longrightarrow y$, $v : y \longrightarrow z$, $w : z \longrightarrow t$.
    - the identities are local units for the composition :

$$1_x \star_0 u \;=\; u \;=\; u \star_0 1_y.$$

    for every 0-cells x, y and 1-cell $u : x \longrightarrow y$.

- Monoid $M(., 1_M) \leftrightarrow$ category $\mathbf{M}$ with one 0-cell $\bullet$:
  - 1-cells of $\mathbf{M}(\bullet, \bullet)$ are elements of M,
  - $1_\bullet \leftrightarrow 1_M$, composition $u \star_0 v$ in $\mathbf{M}(\bullet, \bullet) \leftrightarrow$ product $u.v$ in M.

Two dimensional categories and polygraphs

- A **functor** $F : \mathbf{C} \to \mathbf{D}$ is a data made of a map $F_0 : \mathbf{C}_0 \to \mathbf{D}_0$ and, for every 0-cells x and y of $\mathbf{C}$, a map

$$F_{x,y} \; : \; \mathbf{C}(x,y) \;\to\; \mathbf{D}(F_0(x), F_0(y)),$$

  such that

  - for every 0-cells x, y and z and every 1-cells $u : x \to y$ and $v : y \to z$ of $\mathbf{C}$,

$$F_{x,z}(u \star_0 v) \;=\; F_{x,y}(u) \star_0 F_{y,z}(v),$$

  - for every 0-cell x of $\mathbf{C}$,

$$F_{x,x}(1_x) \;=\; 1_{F_0(x)}.$$

- A **1-polygraph** is a directed graph (Σ_0, Σ_1) :

$$\Sigma_0 \underset{t_0}{\overset{s_0}{\leftleftarrows}} \Sigma_1$$

  - given by a set Σ_0 of **0-cells**, a set Σ_1 of **1-cells**,
  - maps s_0 and t_0 sending a 1-cell x on its **source** $s_0(x)$ and its **target** $t_0(x)$.

Two dimensional polygraphs and polygraphs

- The **free category** Σ_1^* generated by a 1-polygraph (Σ_0, Σ_1) :
  - objects are the 0-cells in Σ_0,
  - for any 0-cells p and q, the elements of $\Sigma_1^*(p,q)$ are paths in (Σ_0, Σ_1) :

$$p \xrightarrow{x_1} p_1 \xrightarrow{x_2} p_2 \xrightarrow{x_3} \dots \xrightarrow{x_{n-1}} p_{n-1} \xrightarrow{x_n} q$$

  - the composition is the concatenation of paths,
  - the identity on a 0-cell p is the empty path with source and target p.
- A 1-polygraph Σ **generates** a category $\mathbf{C}$ if
  - Σ has the same 0-cells as $\mathbf{C}$,
  - for every 0-cells x and y of $\mathbf{C}$, the map

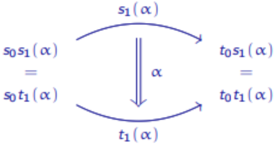
$$\Sigma^*(x,y) \longrightarrow \mathbf{C}(x,y)$$

    is surjective.

Two dimensional polygraphs and polygraphs

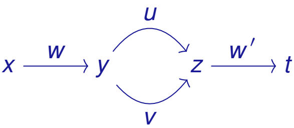
- A **2-polygraph** Σ is a triple $(\Sigma_0, \Sigma_1, \Sigma_2)$, where
  - (Σ_0, Σ_1) is a 1-polygraph,
  - Σ_2 is a **cellular extension** of the free category Σ_1^* :

$$\Sigma_0 \underset{t_0}{\overset{s_0}{\leftleftarrows}} \Sigma_1^* \underset{t_1}{\overset{s_1}{\leftleftarrows}} \Sigma_2$$



  - The elements Σ_2 are called the **2-cells** of Σ, or the **rewriting rules** of Σ.

- A **congruence** on a category $\mathbf{C}$ is an equivalence relation $\equiv$ on parallel 1-cells of $\mathbf{C}$ that is compatible with the composition of $\mathbf{C}$:



of $\mathbf{C}$ such that $u \equiv v$, we have $wuw' \equiv wvw'$.

Two dimensional polygraphs and polygraphs

- If Γ is a cellular extension of $\mathbf{C}$, the congruence $\equiv_\Gamma$ **generated** by Γ is the smallest congruence relation such that, if $\gamma : u \Rightarrow v$ is in Γ, then $u \equiv_\Gamma v$.

- The **quotient** of $\mathbf{C}$ by Γ is the category $\mathbf{C}/\Gamma$:

  - the 0-cells of $\mathbf{C}/\Gamma$ are the ones of $\mathbf{C}$,

  - for every 0-cells x, y of $\mathbf{C}$, the set $\mathbf{C}/\Gamma(x, y)$ is the quotient of $\mathbf{C}(x, y)$ by the restriction of $\equiv_\Gamma$.

- The category $\overline{\Sigma}$ **presented** by a 2-polygraph Σ is the category

$$\overline{\Sigma} = \Sigma_1^*/\Sigma_2.$$

- A **presentation** of a category $\mathbf{C}$ is a 2-polygraph Σ such that $\mathbf{C} \simeq \overline{\Sigma}$.
  - the 1-cells of Σ : **generating 1-cells of $\mathbf{C}$**, or **generators of $\mathbf{C}$**,
  - the 2-cells of Σ : **generating 2-cells of $\mathbf{C}$**, or **relations of $\mathbf{C}$**.

- 2-polygraphs are **Tietze-equivalent** if they present the same category.

Two dimensional polygraphs and polygraphs

Example.

The **plactic monoid** $\mathbf{P}_n$ of rank n is presented by the 2-polygraph $\mathrm{Knuth}_2(n)$:

- set of 1-cells : $[n] := \{1 < \ldots < n\}$,
- 2-cells are the **Knuth relations** :

$$\{ \, zxy \overset{\eta_{x,y,z}}{\Longrightarrow} xzy \mid 1 \leq x \leq y < z \leq n \, \} \cup \{ \, yzx \overset{\varepsilon_{x,y,z}}{\Longrightarrow} yxz \mid 1 \leq x < y \leq z \leq n \, \}.$$

- (Schensted, 61, '70), (Knuth, '70) : **Young tableaux** and insertions.
- (Lascoux, Schützenberger, '81) : theory of **symmetric polynomials**
  - first correct proof of the Littelwood–Richardson rule
- representations of finite-dimensional complex semisimple Lie algebras
  - **Kashiwara crystal theory**
  - **Littelmann path model**
  - Classification of plactic monoids in **classical types** and **exceptional ones**.

Two dimensional categories and polygraphs

- A **2-category** is a data $\mathbf{C}$ made of :
  - a set $\mathbf{C}_0$ of **0-cells** of $\mathbf{C}$,
  - for every 0-cells x, y, a category $\mathbf{C}(x, y)$,
    - whose 0-cells and 1-cells are called the **1-cells** and the **2-cells** from x to y of $\mathbf{C}$.
  - for every 0-cells x, y, z, a functor
  $$\star_0^{x,y,z} \; : \; \mathbf{C}(x,y) \times \mathbf{C}(y,z) \; \longrightarrow \; \mathbf{C}(x,z),$$
  - for every 0-cell x, a specified 0-cell 1_x of the category $\mathbf{C}(x, x)$,

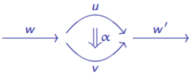
such that

- Associativity : for every 0-cells x, y, z and t :
$$\star_0^{x,z,t} \circ (\star_0^{x,y,z} \times \mathrm{Id}_{\mathbf{C}(z,t)}) \; = \; \star_0^{x,y,t} \circ (\mathrm{Id}_{\mathbf{C}(x,y)} \times \star_0^{y,z,t}),$$
- Identities axiom : for every 0-cells x and y :
$$\star_0^{x,x,y} \circ (1_x \times \mathrm{Id}_{\mathbf{C}(x,y)}) \; = \; \mathrm{Id}_{\mathbf{C}(x,y)} \; = \; \star_0^{x,y,y} \circ (\mathrm{Id}_{\mathbf{C}(x,y)}, 1_y).$$

Two dimensional categories and polygraphs

The **free 2-category** Σ^* over a 2-polygraph Σ :

- the 0-cells of Σ^* are the ones of Σ,
- for every 0-cells x and y of Σ, the category $\Sigma_2^*(x, y)$ is defined as
  - the free category over the 1-polygraph whose
    - 0-cells are the 1-cells from x to y of Σ_1^*,
    - 1-cells are the



with α in Σ_2 and w and w' in Σ_1^*,

  - quotiented by the congruence generated by the cellular extension

$$\alpha w s(\beta) \star_1 t(\alpha) w \beta \equiv s(\alpha) w \beta \star_1 \alpha w t(\beta),$$

for α and β in Σ_2 and w in Σ_1^*.

Plan :

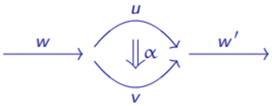
1. **Two dimensional categories and polygraphs**

2. **Rewriting properties of 2 polygraphs**

Rewriting properties of 2-polygraphs

Let $\Sigma = (\Sigma_0, \Sigma_1, \Sigma_2)$ be a 2-polygraph.

- A **rewriting step** of Σ is a 2-cell of the free 2-category Σ_2^* :



  where α is 2-cell of Σ_2 and w, w' are 1-cells of Σ_1^*.

- A **rewriting sequence** of Σ :

$$w_1 \Longrightarrow w_2 \Longrightarrow \cdots \Longrightarrow w_n \Longrightarrow \cdots .$$

- w **rewrites** into w' : Σ has a non-empty rewriting sequence from w to w'.

- w is a **normal form** : Σ has no rewriting step with source w.

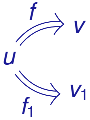
- w' is a **normal form of** w : w' is a normal form and w rewrites into w'.

- Σ **terminates** if it has no infinite rewriting sequences.

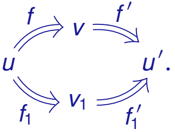
Rewriting properties of 2-polygraphs

Let $\Sigma = (\Sigma_0, \Sigma_1, \Sigma_2)$ be a 2-polygraph.

- **Branching** of Σ is a pair (f, f_1) of 2-cells of Σ_2^* with a common source :



  - **Local branching** : f and f_1 are rewriting steps.
- A branching is **confluent** :



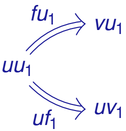
- Σ is **confluent** : all of its branchings are confluent.
  - **locally confluent** : all of its local branchings are confluent.

Newman's Lemma. The local confluence property and the confluence property are equivalent for terminating 2-polygraphs.

Rewriting properties of 2-polygraphs

Let $\Sigma = (\Sigma_0, \Sigma_1, \Sigma_2)$ be a 2-polygraph.

- Σ is **convergent** if it terminates and it is confluent.
  - every 1-cell w in Σ_1^* has a unique normal form $\widehat{w}$:
    - $w = w'$ in $\overline{\Sigma}$ if, and only if, $\widehat{w} = \widehat{w'}$ holds in Σ_1^*.
- **Local branchings** of Σ :
  - **aspherical** branchings : shape (f, f) with source u and target (v, v).
  - **Peiffer** branchings :



  - **overlapping** branchings : remaining local branchings.
- Local branchings are ordered by the order $\sqsubseteq$ generated by

$$(f, f_1) \sqsubseteq (ufv, uf_1 v).$$

- **Critical branching** : overlapping local branching minimal for $\sqsubseteq$.

Critical pair theorem. A 2-polygraph is locally confluent if, and only if, all its critical branchings are confluent.

Example. The Knuth presentation $\mathrm{Knuth}_2(2)$ of the plactic monoid $\mathbf{P}_2$:

- 1-cells : 1, 2,
- 2-cells : $\eta_{1,1,2} : 211 \Rightarrow 121$, $\varepsilon_{1,2,2} : 221 \Rightarrow 212$.
- This presentation is
  - terminating,
  - convergent :

$$2211 \overset{2\eta_{1,1,2}}{\underset{\varepsilon_{1,2,2}1}{\rightrightarrows}} 2121.$$

Plan :

1. **Two dimensional categories and polygraphs**

2. **Rewriting properties of** 2 **polygraphs**

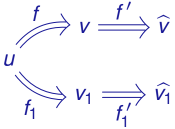
   - **The Knuth–Bendix's completion**

Rewriting properties of 2-polygraphs

Consider a terminating 2-polygraph Σ, with a total termination order $\leq$.

The **Knuth–Bendix's completion** of $\Sigma \rightsquigarrow$ 2-polygraph $\mathcal{KB}(\Sigma)$:

- We start with $\mathcal{KB}(\Sigma) = \Sigma$ and the set $\mathcal{CB}$ of critical branchings of Σ.
  - If $\mathcal{CB}$ is empty, then the procedure stops.
  - Otherwise, we pick a branching (f, f_1) with source u :



- If $\widehat{v} = \widehat{v_1}$, then (f, f_1) is confluent and we pass to next critical branching,
- If $\widehat{v} > \widehat{v_1}$, we add $\alpha : \widehat{v} \Rightarrow \widehat{v_1}$ to $\mathcal{KB}(\Sigma)$ and all the new critical branchings created by α to $\mathcal{CB}$,
- If $\widehat{v} < \widehat{v_1}$, we add $\alpha : \widehat{v_1} \Rightarrow \widehat{v}$ to $\mathcal{KB}(\Sigma)$ and all the new critical branchings created by α to $\mathcal{CB}$,
- we remove (f, f_1) from $\mathcal{CB}$ and restart from the beginning.

Theorem. (Knuth–Bendix, 70). $\mathcal{KB}(\Sigma)$ is a convergent presentation of $\overline{\Sigma}$. Moreover, $\mathcal{KB}(\Sigma)$ is finite if, and only if, Σ is finite and the Knuth–Bendix's completion procedure halts.

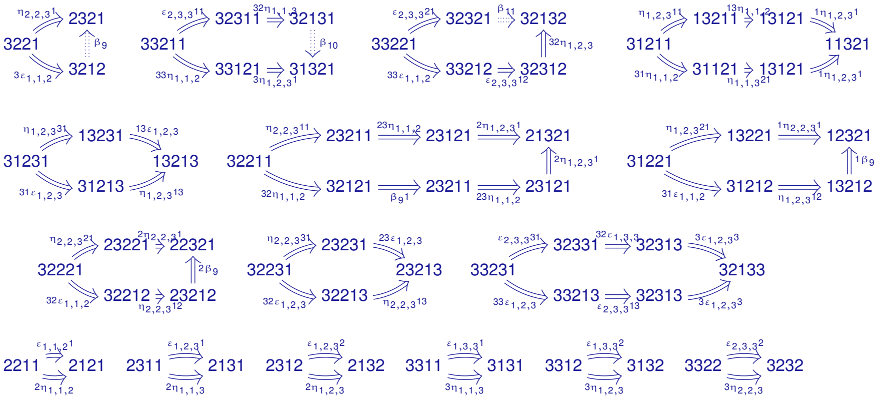
Rewriting properties of 2-polygraphs

Example. Consider the Knuth presentation $\mathrm{Knuth}_2(3)$ of $\mathbf{P}_3$ whose 2-cells are

$$211 \overset{\eta_{1,1,2}}{\Longrightarrow} 121, \quad 311 \overset{\eta_{1,1,3}}{\Longrightarrow} 131, \quad 312 \overset{\eta_{1,2,3}}{\Longrightarrow} 132, \quad 322 \overset{\eta_{2,2,3}}{\Longrightarrow} 232,$$

$$221 \overset{\varepsilon_{1,1,2}}{\Longrightarrow} 212, \quad 231 \overset{\varepsilon_{1,2,3}}{\Longrightarrow} 213, \quad 331 \overset{\varepsilon_{1,3,3}}{\Longrightarrow} 313, \quad 332 \overset{\varepsilon_{2,3,3}}{\Longrightarrow} 323.$$

This presentation admits the following critical branchings :

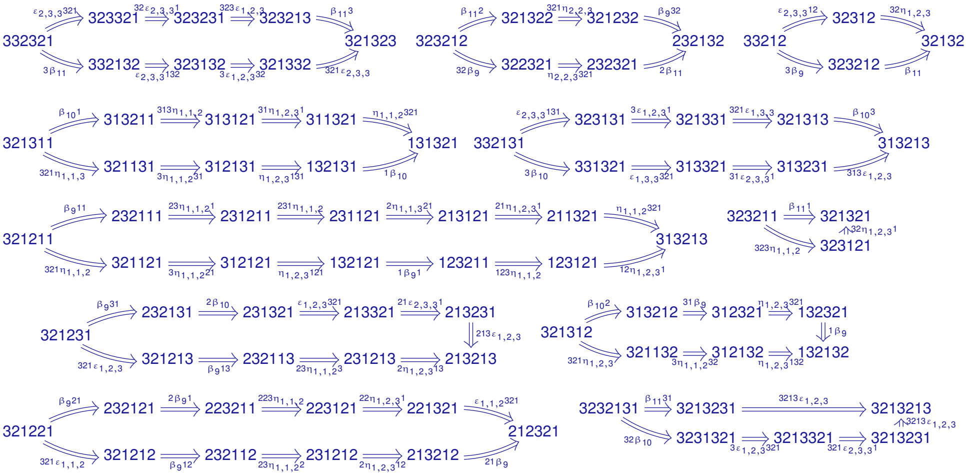


Rewriting properties of 2-polygraphs

By Knuth–Bendix's completion procedure, we add the following 2-cells

$$3212 \overset{\beta_9}{\Longrightarrow} 2321, \quad 32131 \overset{\beta_{10}}{\Longrightarrow} 31321, \quad 32321 \overset{\beta_{11}}{\Longrightarrow} 32132.$$

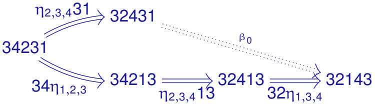
Again using these new 2-cells, we obtain the following critical branchings

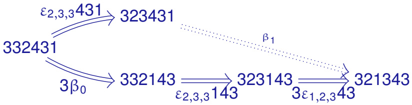


Rewriting properties of 2-polygraphs

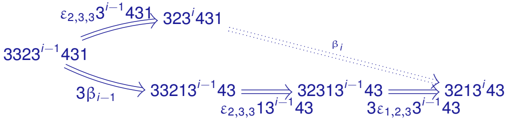
Theorem. (Kubat-Okninski, 11). For $n > 3$, there is no finite completion of $\mathcal{KB}(\mathrm{Knuth}_2(n))$ compatible with the lexicographic order.

Skech of proof.
Prove by induction that $\mathrm{Knuth}_2(4)$ does not admit a finite completion.





Suppose that there exists a rule $\beta_{i-1} : 323^{i-1}431 \Rightarrow 3213^{i-1}43$ added after the i-th step of completion. Then :



Question. Does the plactic monoid admit a finite convergent presentation ?

Plan :

1. **Two dimensional categories and polygraphs**

2. **Rewriting properties of 2 polygraphs**

   - **The Knuth–Bendix's completion**

3. **Column presentation for the plactic monoid of type A**

Column presentation of the plactic monoid of type A

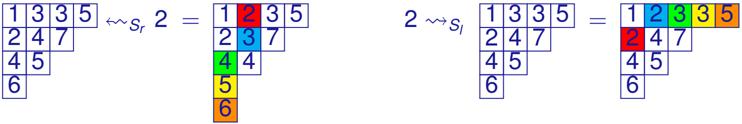
- **(Young) tableaux** :

t =

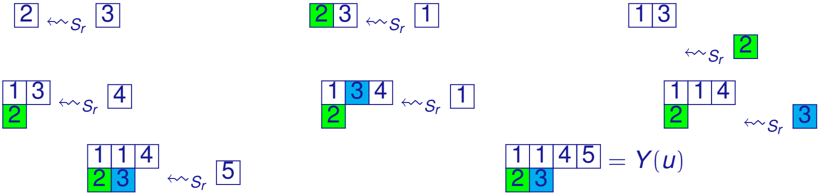
1	1	1	2	4	4	4
2	2	3	3	5	7	
4	5	5	6			
6	8					

$R_{col}(t) = 6421\ 8521\ 531\ 632\ 54\ 74\ 4$

- **Schensted's insertions** :



- $u = 231415$



Column presentation of the plactic monoid of type A

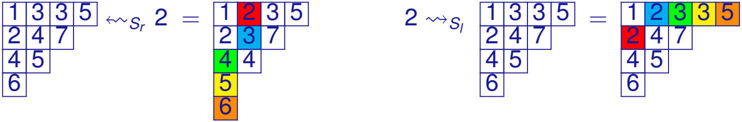
- **(Young) tableaux** :

t =

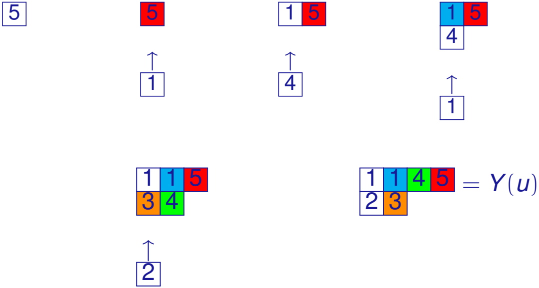
1	1	1	2	4	4	4
2	2	3	3	5	7	
4	5	5	6			
6	8					

$R_{col}(t) = 6421\ 8521\ 531\ 632\ 54\ 74\ 4$

- **Schensted's insertions** :



- $u = 231415$



Column presentation of the plactic monoid of type A

- **(Young) tableaux** :

$$t = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 4 & 4 & 4 \\ \hline 2 & 2 & 3 & 3 & 5 & 7 \\ \cline{1-6} 4 & 5 & 5 & 6 \\ \cline{1-4} 6 & 8 \\ \cline{1-2} \end{array} \qquad R_{col}(t) = 6421\ 8521\ 531\ 632\ 54\ 74\ 4$$

- **Schensted's insertions** :

$$\begin{array}{|c|c|c|c|}\hline 1 & 3 & 3 & 5 \\ \hline 2 & 4 & 7 \\ \cline{1-3} 4 & 5 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array} \leftsquigarrow_{S_r} 2 = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 5 \\ \hline 2 & 3 & 7 \\ \cline{1-3} 4 & 4 \\ \cline{1-2} 5 \\ \cline{1-1} 6 \\ \cline{1-1} \end{array} \qquad 2 \rightsquigarrow_{S_l} \begin{array}{|c|c|c|c|}\hline 1 & 3 & 3 & 5 \\ \hline 2 & 4 & 7 \\ \cline{1-3} 4 & 5 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array} = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 3 & 5 \\ \hline 2 & 4 & 7 \\ \cline{1-3} 4 & 5 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array}$$

- $u = 231415$

$$((((((\emptyset \leftsquigarrow_{S_r} 2) \leftsquigarrow_{S_r} 3) \leftsquigarrow_{S_r} 1) \leftsquigarrow_{S_r} 4) \leftsquigarrow_{S_r} 1) \leftsquigarrow_{S_r} 5)$$

$$= \begin{array}{|c|c|c|c|}\hline 1 & 1 & 4 & 5 \\ \hline 2 & 3 \\ \cline{1-2} \end{array}$$

$$= \ (2 \rightsquigarrow_{S_l} (3 \rightsquigarrow_{S_l} (1 \rightsquigarrow_{S_l} (4 \rightsquigarrow_{S_l} (1 \rightsquigarrow_{S_l} (5 \rightsquigarrow_{S_l} \emptyset))))))$$

Column presentation of the plactic monoid of type A

- The juxtaposition of two **columns** $u = x_p \dots x_1$ and $v = y_q \dots y_1$

$$\begin{array}{|c|c|} \hline x_1 & y_1 \\ \vdots & \vdots \\ \hline x_q & y_q \\ \hline \vdots & \\ \cline{1-1} x_p & \\ \cline{1-1} \end{array}$$

  can form a Young tableau.

- Denote $u \stackrel{\times}{\ } v$ in other cases :

$$u = \begin{array}{|c|} \hline 3 \\ \hline 5 \\ \hline 7 \\ \hline \end{array} \quad \text{and} \quad v = \begin{array}{|c|} \hline 4 \\ \hline 6 \\ \hline 8 \\ \hline 9 \\ \hline \end{array} \quad \text{or} \quad u = \begin{array}{|c|} \hline 3 \\ \hline 6 \\ \hline 7 \\ \hline 9 \\ \hline \end{array} \quad \text{and} \quad v = \begin{array}{|c|} \hline 4 \\ \hline 5 \\ \hline 8 \\ \hline \end{array}.$$

Lemma. Suppose $u \stackrel{\times}{\ } v$. The tableau $Y(uv)$ consists of at most two columns. Moreover, if $Y(uv)$ contains exactly two columns, the left column contains more elements than u.

- For $u \stackrel{\times}{\ } v$, define

$$c_u c_v \overset{\alpha_{u,v}}{\Longrightarrow} c_w c_{w'}$$

  such that
  - w and w' are the columns of $Y(uv)$ if it consists of two columns,
  - $w = uv$ and w' is empty, otherwise.

Column presentation of the plactic monoid of type A

Theorem. (Cain-Gray-Malheiro,14). The 2-polygraph $\mathrm{Col}_2(n)$:

- 1-cells : column generators,
- 2-cells : $c_u c_v \overset{\alpha_{u,v}}{\Longrightarrow} c_w c_{w'}$, for $u \overset{\times}{\;} v$,

is a finite convergent presentation of the plactic monoid $\mathbf{P}_n$.

Example. For $n = 2$,

$$\boxed{1},\qquad \boxed{2},\qquad \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}.$$

We have

$$\boxed{2} \leftsquigarrow_{S_r} \boxed{1} = \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \qquad \boxed{1} \leftsquigarrow_{S_r} \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} = \begin{array}{|c|c|}\hline 1 & 1\\\hline 2 \\\cline{1-1}\end{array} \qquad \boxed{2} \leftsquigarrow_{S_r} \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} = \begin{array}{|c|c|}\hline 1 & 2\\\hline 2 \\\cline{1-1}\end{array}$$

Then

$$c_2 c_1 \overset{\alpha_{2,1}}{\Longrightarrow} c_{21} \qquad c_1 c_{21} \overset{\alpha_{1,21}}{\Longrightarrow} c_{21} c_1 \qquad c_2 c_{21} \overset{\alpha_{2,21}}{\Longrightarrow} c_{21} c_2.$$

Hence

$$\mathrm{Col}_2(2) = \langle\; c_1, c_2, c_{21} \mid c_2 c_1 \overset{\alpha_{2,1}}{\Longrightarrow} c_{21},\; c_1 c_{21} \overset{\alpha_{1,21}}{\Longrightarrow} c_{21} c_1,\; c_2 c_{21} \overset{\alpha_{2,21}}{\Longrightarrow} c_{21} c_2 \;\rangle.$$

Plan :

1. **Two dimensional categories and polygraphs**

2. **Rewriting properties of** 2 **polygraphs**

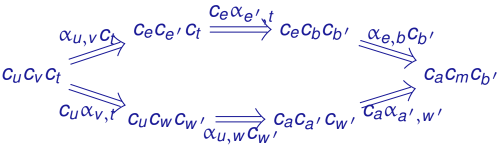
   - **The Knuth–Bendix's completion**

3. **Column presentation for the plactic monoid of type A**

4. **Coherence**

Coherence

- **Coherent presentation** :
  - a presentation of the category,
    - generators
    - rules
  - globular **homotopy generators** : the relations amongst the relations
  - **Squier's theorem**
- Column coherent presentation of the plactic monoid, (H., Malbos, '16) :
  - generators : columns
  - rules : $\alpha_{u,v} : c_u c_v \Rightarrow c_w c_{w'}$
  - homotopy generators :



- Plactic monoids of classical and exceptional types : convergent presentations ? coherent presentations ?