Operadic rewriting and Koszulness

Isaac Ren
joint work with Philippe Malbos

ENS de Lyon - Université de Lyon

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1. Overview

2. Symmetric and shuffle operads

3. Gröbner bases for shuffle operads

4. Polygraphic rewriting in shuffle operads

5. Higher dimensional polygraphs and Koszulness

Overview

Why algebraic rewriting?

- Rewriting is a **combinatorial** theory of equivalence [Newman 1942]
- Algebraic rewriting is a combinatorial theory of **congruence**
  - In computer algebra: **ideal membership**, **resolutions**, **homological properties**

  Examples: monoids, commutative algebras, associative algebras [Shirshov 1962, Bergman 1978, Bokut' 1994, Mora 1994], higher categories [Street 1976, Burroni 1993], operads [Dotsenko-Khoroshkin 2010].

  - In constructive mathematics: **cofibrant replacements**

  Examples: associative and path algebras [Anick 1986, Anick-Green 1987, Guiraud-Hoffbeck-Malbos 2019], monoids [Kobayashi 1990, Brown 1992], categories [Guiraud-Malbos 2009], operads [Dotsenko-Khoroshkin 2013].

Two ways of doing rewriting:

- **Gröbner bases**:
  - we orient relations using a monomial order,
  - we formulate confluence algebraically.
- **Polygraphs**:
  - we work in a higher dimensional setting,
  - we interpret rewriting properties by oriented syzygies:

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd} \quad \Rrightarrow \quad \begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd}$$

Our goal is to mix the two approaches.

Theorem [Malbos-R. 2020]

An operad with a quadratic convergent presentation is Koszul.

Symmetric and shuffle operads

If associative algebras are a linear version of words, then shuffle operads are a linear version of planar trees.

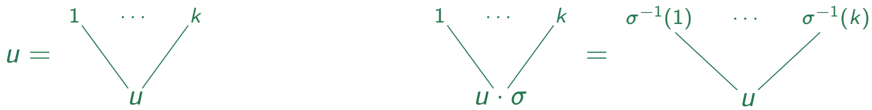
Symmetric operads

- The category $\mathfrak{S}\mathsf{Coll}$ of **symmetric collections** is the presheaf category on Fin, the category of finite nonempty sets with bijections, with values in Vect, the category of vector spaces over $\mathbf{k}$.
  - A symmetric collection V is determined by $V(k) := V(\{1,\dots,k\})$ for $k \geq 1$, with a right action by $\mathfrak{S}_k$. An element of $V(k)$ is of **arity** k.
- The **symmetric composition** of two collections V, W is

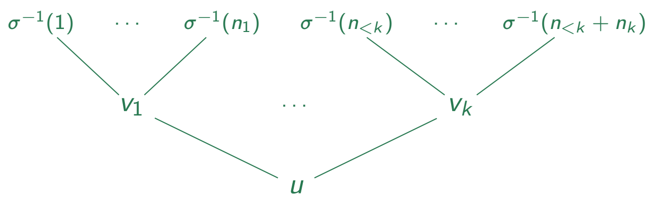
$$V \circ_{\mathfrak{S}} W(I) = \bigoplus_{k\geq 1} V(k) \otimes_{\mathbf{k}[\mathfrak{S}_k]} \left(\bigoplus_{f : I \twoheadrightarrow \{1,\dots,k\}} W(f^{-1}\{1\}) \otimes \cdots \otimes W(f^{-1}\{k\}) \right)$$

  where $I \in \mathsf{Fin}$ and f is a surjection. The unit for this composition is $\mathbb{1} := (\mathbf{k}, 0, \dots)$.
- $(\mathfrak{S}\mathsf{Coll}, \circ_{\mathfrak{S}}, \mathbb{1})$ is a monoidal category.
- The category $\mathfrak{S}\mathsf{Op}$ of **symmetric operads** is the category of internal monoids in $(\mathfrak{S}\mathsf{Coll}, \circ_{\mathfrak{S}}, \mathbb{1})$.

We can represent an element $u \in V(k)$ and its image by $\sigma \in \mathfrak{S}_k$ as planar trees with indexed inputs:



Then, for $u \in V(k), v_1 \in V(n_1), \ldots, v_k \in V(n_k)$, $n_{<k} = n_1 + \cdots + n_{k-1}$, and $\sigma \in \mathfrak{S}_{n_{<k}+n_k}$, we have



$$\begin{aligned}
&\in V(k) \otimes_{\mathfrak{S}} \Big(V(\sigma^{-1}\{1, \ldots, n_1\}) \otimes \cdots \otimes V(\sigma^{-1}\{n_{<k}, \ldots, n_{<k} + n_k\} \Big) \\
&\subseteq (V \circ_{\mathfrak{S}} V)(n_1 + \cdots + n_k).
\end{aligned}$$

Elements of a symmetric operad can be represented as linear combinations of planar trees.

Example: symmetric operad Lie

- The symmetric operad Lie is generated by one antisymmetric **operation** μ of **arity** 2, satisfying the **Jacobi relation**

$$\mu(\mu(1,2),3) + \mu(\mu(2,3),1) + \mu(\mu(3,1),2) = 0.$$

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$$\begin{array}{ccccc} 1 & & 2 & & \\ & \diagdown\ \ \diagup & & & \\ & \mu & & & 3 \\ & & \diagdown & \diagup & \\ & & \mu & & \end{array} + \begin{array}{ccccc} 2 & & 3 & & \\ & \diagdown\ \ \diagup & & & \\ & \mu & & & 1 \\ & & \diagdown & \diagup & \\ & & \mu & & \end{array} + \begin{array}{ccccc} 3 & & 1 & & \\ & \diagdown\ \ \diagup & & & \\ & \mu & & & 2 \\ & & \diagdown & \diagup & \\ & & \mu & & \end{array} = 0.$$

- Compare with

$$[[x_1, x_2], x_3] + [[x_2, x_3], x_1] + [[x_3, x_1], x_2] = 0.$$

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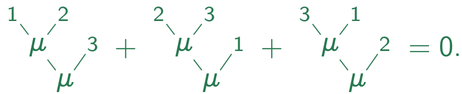


- Compare with

$$[[x_1, x_2], x_3] + [[x_2, x_3], x_1] + [[x_3, x_1], x_2] = 0.$$

- Due to the symmetric actions, there is no known way to do algebraic rewriting in symmetric operads.
  - This motivates the study of **shuffle operads** [Dotsenko-Khoroshkin 2010].

Shuffle operads [Dotsenko-Khoroshkin 2010]

- The category Coll of **(non-symmetric) collections** is the presheaf category on Ord, the category of finite nonempty ordered sets with order-preserving bijections, with values in Vect.
  - A collection V is determined by $V(k) := V(\{1 < \cdots < k\})$ for $k \geq 1$, whose elements are of **arity** k.
- The **shuffle composition** of two collections V, W is

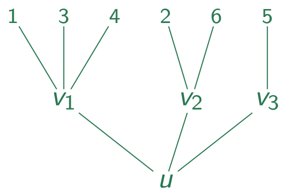
$$V \circ_{\text{Ш}} W(I) = \bigoplus_{k \geq 1} V(k) \otimes \left(\bigoplus_{f : I \twoheadrightarrow \{1,\ldots,k\}} W(f^{-1}\{1\}) \otimes \cdots \otimes W(f^{-1}\{k\}) \right)$$

  where $I \in \mathrm{Ord}$ and f is a **shuffle surjection**, that is, $\min f^{-1}\{1\} < \cdots < \min f^{-1}\{k\}$. The unit for this composition is also $\mathbb{1} := (\mathbf{k}, 0, \ldots)$.
- $(\mathrm{Coll}, \circ_{\text{Ш}}, \mathbb{1})$ is a monoidal category.
- The category ШOp of **shuffle operads** is the category of internal monoids in $(\mathrm{Coll}, \circ_{\text{Ш}}, \mathbb{1})$.

An example of a shuffle surjection is the surjection $f : \{1, \dots, 6\} \to \{1, 2, 3\}$ defined by

$$f(1) = f(3) = f(4) = 1, \qquad f(2) = f(6) = 2, \qquad f(5) = 3.$$

For $u \in V(3)$, $v_1 \in V(3)$, $v_2 \in V(2)$, $v_3 \in V(1)$, we then have



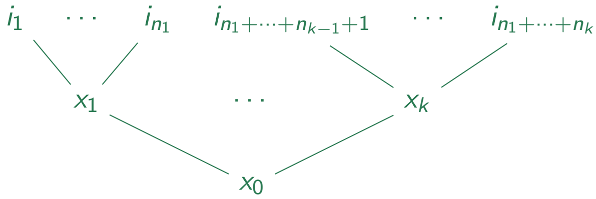
$$\in V(3) \otimes \Big(V(f^{-1}\{1\}) \otimes V(f^{-1}\{2\}) \otimes V(f^{-1}\{3\}) \Big)$$

$$\subseteq (V \circ_{\text{Ш}} V)(6).$$

We will denote this element by $(u \mid_f v_1\, v_2\, v_3)$ or $(u \mid \vec{v})$.

Tree monomials

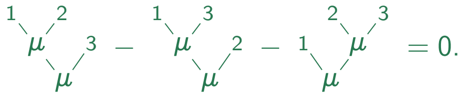
- Let $X = (X(k))_{k\geq 1}$ such that $X(k)$ is a basis of $V(k)$ for every $k \geq 1$. In terms of planar trees, the collection $V \circ_{\text{Ш}} V$ has a basis of planar trees



  - For $j \in \{1, \ldots, k\}$, the **inputs** of x_j are $\{i_{n_1+\cdots+n_{j-1}+1} < \cdots < i_{n_1+\cdots+n_j}\}$.
  - The inputs of x_0 are $\{i_1 < i_{n_1+1} < \cdots < i_{n_1+\cdots+n_{k-1}+1}\}$.
- By iterating this tree construction, we get the **free shuffle operad on** X, denoted by $X^{\text{Ш}}$ spanned by **tree monomials**. We refer to elements of $X^{\text{Ш}}(k)$ as **polynomials of arity** k.

Example: shuffle operad $\mathrm{Lie}^\flat$

The shuffle operad $\mathrm{Lie}^\flat$ is generated by one operation μ of arity 2, and satisfies the **shuffle Jacobi relation**



Bimodules over operads

- Given a shuffle operad P, a P-**bimodule** is a collection M equipped with linear maps

$$\lambda : P(k) \otimes P(f^{-1}\{1\}) \otimes \cdots \otimes A(f^{-1}\{i\}) \otimes \cdots \otimes P(f^{-1}\{k\}) \to A(I) \qquad \text{(crossed left action)}$$
$$\rho : A(k) \otimes P(f^{-1}\{1\}) \otimes \cdots \otimes P(f^{-1}\{k\}) \to A(I), \qquad \text{(right action)}$$

  for all $k \geq 1$, $I \in \mathbf{Ord}$, and $f : \{1, \ldots, k\} \twoheadrightarrow I$ a shuffle surjection.

- An **ideal** of a shuffle operad P is a sub-P-bimodule of P, seen as a P-bimodule.

Forgetting symmetric actions

- There is a forgetful functor $\mathsf{Ord} \to \mathsf{Fin}$ that forgets the order. This induces forgetful functors

$$-^u : \mathfrak{S}\mathsf{Coll} \to \mathsf{Coll} \qquad \text{and} \qquad -^u : \mathfrak{S}\mathsf{Op} \to \text{Ш}\mathsf{Op}.$$

- Given a symmetric operad P, the shuffle operad P^u preserves all of the information of the symmetric actions.
- [Dotsenko-Khoroshkin 2013] Given a symmetric operad P, denote by $H^Q(P)$ its **Quillen homology**, and by $H^Q(P^u)$ the Quillen homology of P^u. Recall that these are sequences of symmetric or non-symmetric collections. Then

$$H^Q(P)^u \simeq H^Q(P^u).$$

Quadratic operads

- An operad is **quadratic** if it is presented by generators and relations such that the relations are all of homogeneous weight 2.

Koszulness

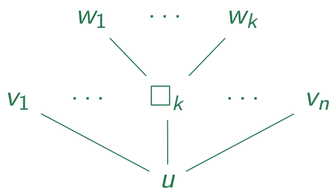
- A quadratic operad is **Koszul** if its Quillen homology is concentrated on the diagonal, that is, its n^{th} homology group is concentrated in weight n.
- In particular, the **cobar construction** on the **Koszul dual coooperad** of a Koszul operad is a minimal model of the operad.
- A quadratic symmetric operad is Koszul if its shuffle version is Koszul.

Gröbner bases for shuffle operads

With the planar tree interpretation, we can define contexts:

Contexts

- A **context of inner arity** k is a tree monomial $C[-]$ of the form



  where $\square_k$ is a symbol of arity k and $u, \vec{v}, \vec{w}$ are tree monomials.

- Given a polynomial $f = \sum \lambda_i u_i$ of arity k, we define the polynomial $C[f] := \sum \lambda_i C[u_i]$.

Monomial orders

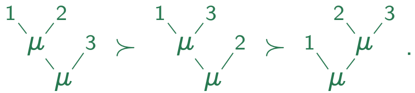
- A **monomial order** is a total order $\prec$ on tree monomials such that for $u \prec v$ two tree monomials and C a context, $C[u] \prec C[v]$.
- Given a polynomial f,
  - its **leading monomial** $\mathrm{lm}(f)$ is the greatest tree monomial that occurs,
  - its **leading coefficient** $\mathrm{lc}(f)$ is the coefficient in front of the leading monomial.

Path-lexicographic order

An example of a monomial order is the **path-lexicographic order**:

- Given a tree monomial, write the path from the root to each input.
- Write the list of inputs from left to right.
- The order is given by lexicographic order on the paths and the list of inputs.

For example,

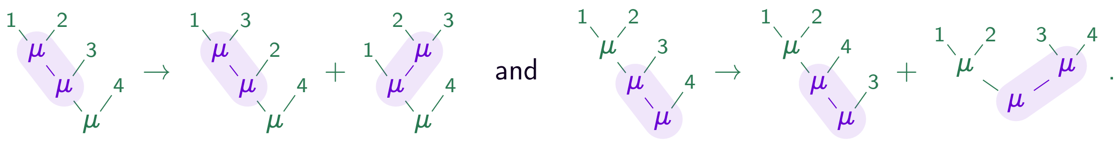


Gröbner bases for operads

- Given two polynomials f and g, if there exists a context C such that $C[\mathrm{lm}(g)] = \mathrm{lm}(f)$, then we define the **reduction** of f by g by

$$f \xrightarrow{C[g]} f - \frac{\mathrm{lc}(f)}{\mathrm{lc}(g)} C[g].$$

- For example, the shuffle Jacobi relation induces the reductions



- A **Gröbner basis** of an ideal I of a free shuffle operad $X^{\text{Ш}}$ is a generating set G such that every nonzero polynomial in I can be reduced by an element of G.

This approach allows us to obtain a homological result on operads:

Theorem [Dotsenko-Khoroshkin 2010, 2013]

A quadratic operad with a quadratic Gröbner basis is Koszul.

Idea of proof.

- Construct a dg-operad called the **inclusion-exclusion operad**, which keeps track of reducible factors.
- This dg-operad is a free resolution of the associated **monomial operad**.
- Obtain a free resolution of the original operad by deformation of the differential.
- This resolution is generated by elements concentrated on the superdiagonal, so the Quillen homology is concentrated on the diagonal, and so the operad is Koszul.

Polygraphic rewriting in shuffle operads

Shuffle 1-operads

- A **shuffle 1-operad** is an internal category in the category ШOp of shuffle operads.

$$P_0 \overset{s_0}{\underset{t_0}{\leftleftarrows}} \xrightarrow{i_1} P_1$$

The elements of P_0 are called **0-cells**, and those of P_1 are called **1-cells**

Shuffle 1-operads

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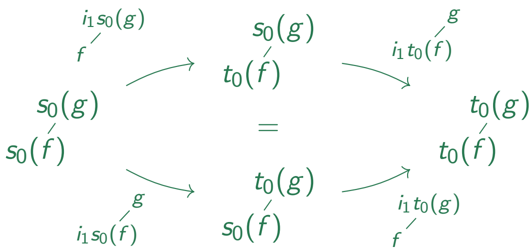
$$\begin{tikzcd} P_0 \arrow[r, "i_1" description] & P_1 \arrow[l, bend right, "s_0"'] \arrow[l, bend left, "t_0"] \end{tikzcd}$$

  The elements of P_0 are called 0-**cells**, and those of P_1 are called 1-**cells**

- The interaction between the internal category structure and the operad structure gives the **linear exchange relation**: for any 1-cells $f : s_0(f) \to t_0(f)$ and $g : s_0(g) \to t_0(g)$, the two paths below are equal:



Shuffle 1-polygraphs

A **shuffle 1-polygraph** is a diagram

$$\begin{tikzcd} X_0^{\text{Ш}} & \\ X_0 \arrow[u] & X_1 \arrow[ul, shift left, "s_0"'] \arrow[ul, shift right, "t_0"] \end{tikzcd}$$

where

- $X_0 = (X_0(k))_{k\geq 1}$ is the indexed set of **generators**
- $X_1 = (X_1(k))_{k\geq 1}$ is the indexed set of **rewriting rules**
- the **source** and **target** maps $s_0, t_0 : X_1 \to X_0^{\text{Ш}}$ are from rewriting rules to the free operad on the generators.

Shuffle 1-polygraphs

A **shuffle 1-polygraph** is a diagram

$$\begin{tikzcd}
X_0^{\mathrm{III}} \arrow[r, "i_1" description] & X_1^{\mathrm{III}} \arrow[l, shift right=2, "s_0"'] \arrow[l, shift left=2, "t_0"] \\
X_0 \arrow[u] & X_1 \arrow[u] \arrow[ul, shift left, "s_0"] \arrow[ul, shift right, "t_0"']
\end{tikzcd}$$

where

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- the **source** and **target** maps $s_0, t_0 : X_1 \to X_0^{\mathrm{III}}$ are from rewriting rules to the free operad on the generators.
- $X^{\mathrm{III}} = (X_0^{\mathrm{III}}, X_1^{\mathrm{III}})$ is the **free shuffle 1-operad** where X_0^{III} is the shuffle operad of 0-cells and X_1^{III} is the shuffle operad of 1-cells.
- The shuffle operad **presented** by X is the coequalizer $\overline{X}$ of $s_0, t_0 : X_1^{\mathrm{III}} \rightrightarrows X_0^{\mathrm{III}}$.

Example: polygraphic presentation of $\mathsf{Lie}^{\flat}$

The shuffle operad $\mathsf{Lie}^{\flat}$ is presented by the shuffle 1-polygraph

$$X_{\mathsf{Lie}^{b}} := \left\langle\; \mu \in X_0(2) \;\middle|\; \alpha : \begin{array}{ccccc} 1 & & 2 & & \\ & \mu & & & 3 \\ & & & \mu & \end{array} \to \begin{array}{ccccc} 1 & & 3 & & \\ & \mu & & & 2 \\ & & & \mu & \end{array} + \begin{array}{ccccc} & & 2 & & 3 \\ 1 & & & \mu & \\ & \mu & & & \end{array} \;\right\rangle.$$

Rewriting systems from 1-polygraphs

Let X be a **left-monomial** 1-polygraph, that is, every source is a tree monomial.

- A **rewriting step** is a 1-cell

$$\lambda C[\alpha] + i_1(b) : \lambda C[u] + b \to \lambda C[a] + b$$

  of X_1^{III}, where $\alpha : u \to a$ is a rewriting rule, C is a context, λ is a nonzero scalar, and b is a polynomial of X_0^{III} such that $C[u] \notin \mathrm{supp}(b)$.

- A **rewriting path** is a sequence of rewriting steps.

- The 1-polygraph X is **terminating** if there are no infinite rewriting paths.

Branchings

- A **branching** is a pair of rewriting paths (f, g) with the same source.
- A **local branching** is a branching (f, g) where f and g are rewriting steps. We classify local branchings as:

additive **multiplicative** **intersecting** **critical**

Confluence

- The 1-polygraph X is (**locally**) **confluent** if, for every (local) branching (f, g), there exist rewriting paths h and k and the confluent diagram

$$\begin{tikzcd} & b \arrow[dr, bend left, "h"] & \\ a \arrow[ur, bend left, "f"] \arrow[dr, bend right, "g"'] & & d \\ & c \arrow[ur, bend right, "k"'] & \end{tikzcd}$$

- The 1-polygraph X is **convergent** if it is confluent and terminating.
- A Gröbner basis is equivalent to a convergent 1-polygraph whose rewriting rules reduce the leading term to the rest.

Cellular extension

Let X be a 1-polygraph.

- A **cellular extension** is an indexed set of **generating 2-cells**

$$\begin{tikzcd} s_0(A) \arrow[rr, bend left, "s_1(A)"] \arrow[rr, bend right, "t_1(A)"'] \arrow[rr, phantom, "\Downarrow A"] & & t_0(A) \end{tikzcd}$$

where $s_0(A), t_0(A)$ are 0-cells and $s_1(A), t_1(A) : a \to b$ are 1-cells of X^{III}.

- Let $\sim$ be the equivalence relation generated by $s_1(A) \sim t_1(A)$ for every element A of the cellular extension. The cellular extension is **acyclic** if the equivalence relation $\sim$ has one equivalence class.

The critical branchings theorem comes from [Knuth-Bendix 1971, Nivat 1972]. The coherent version comes from [Squier 1994, Guiraud-Hoffbeck-Malbos 2019, Malbos-R. 2020].

Theorem (coherent critical branchings)

Let X be a terminating 1-polygraph with a generating 2-cell for each critical branching (f, g):

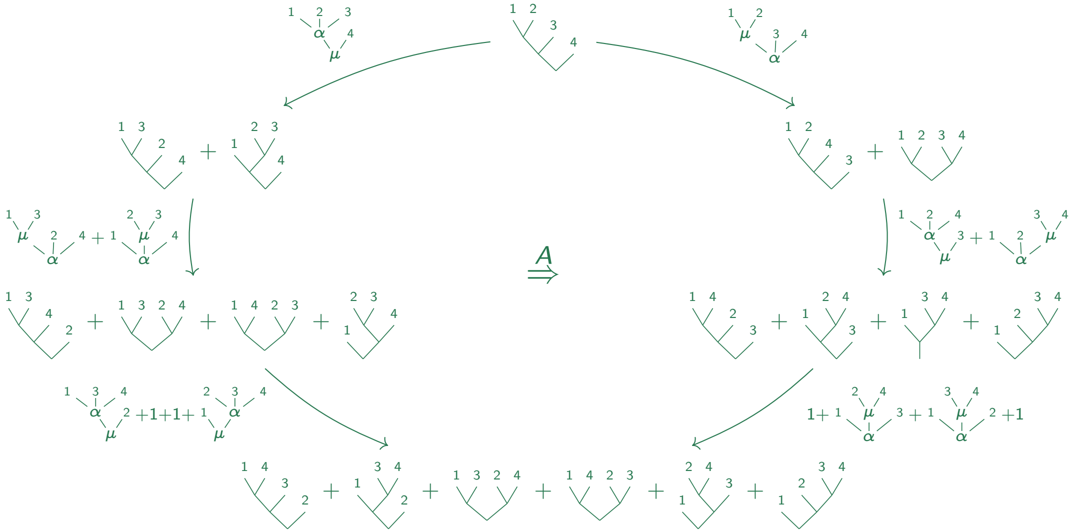
$$\begin{tikzcd} & b \arrow[dr, "h"] \arrow[d, phantom, "\Downarrow A"] & \\ a \arrow[ur, "f"] \arrow[dr, "g"'] & {} & d \\ & c \arrow[ur, "k"'] & \end{tikzcd}$$

Then the cellular extension is acyclic.

We can then consider compositions of generating 2-cells by gluing confluent diagrams: this leads to the notion of **higher dimensional rewriting**.

Example: coherent convergence of $X_{\mathrm{Lie}^\flat}$

The 1-polygraph $X_{\mathrm{Lie}^\flat}$ only has one critical pair and is convergent. The cellular extension will have only one generating 2-cell:



Higher dimensional polygraphs and Koszulness

Shuffle ω-operads

⟩ A **shuffle ω-operad** is an internal (strict) ω-category in $\mathrm{ШOp}$, that is, an object

$$P_0 \overset{s_0}{\underset{t_0}{\leftleftarrows}} \xrightarrow{i_1} P_1 \overset{s_1}{\underset{t_1}{\leftleftarrows}} \xrightarrow{i_2} P_2 \overset{s_1}{\underset{t_1}{\leftleftarrows}} \xrightarrow{i_3} \cdots \overset{s_{n-1}}{\underset{t_{n-1}}{\leftleftarrows}} \xrightarrow{i_n} P_n \overset{s_n}{\underset{t_n}{\leftleftarrows}} \xrightarrow{i_{n+1}} \cdots$$

satisfying globularity, associativity, and identity axioms.

Shuffle ω-polygraphs

〉 The definition of shuffle 1-polygraphs extends to that of **shuffle ω-polygraphs**:

$$
\begin{tikzcd}[column sep=large, row sep=large]
X_0^{\mathrm{III}} \arrow[r, "i_1" description] & X_1^{\mathrm{III}} \arrow[l, shift right=2, "s_0"'] \arrow[l, shift left=2, "t_0"] \arrow[r, "i_2" description] & X_2^{\mathrm{III}} \arrow[l, shift right=2, "s_1"'] \arrow[l, shift left=2, "t_1"] \arrow[r, "i_3" description] & \cdots \arrow[l, shift right=2, "s_1"'] \arrow[l, shift left=2, "t_1"] \arrow[r, "i_n" description] & X_n^{\mathrm{III}} \arrow[l, shift right=2, "s_{n-1}"'] \arrow[l, shift left=2, "t_{n-1}"] \arrow[r, "i_{n+1}" description] & \cdots \arrow[l, shift right=2, "s_n"'] \arrow[l, shift left=2, "t_n"] \\
X_0 \arrow[u] & X_1 \arrow[u] \arrow[ul, shift left, "s_0"'] \arrow[ul, shift right, "t_0"] & X_2 \arrow[u] \arrow[ul, shift left, "s_1"'] \arrow[ul, shift right, "t_1"] & \cdots \arrow[ul, shift left, "s_1"'] \arrow[ul, shift right, "t_1"] & X_n \arrow[u] \arrow[ul, shift left, "s_{n-1}"'] \arrow[ul, shift right, "t_{n-1}"] & \cdots \arrow[ul, shift left, "s_n"'] \arrow[ul, shift right, "t_n"]
\end{tikzcd}
$$

〉 An ω-polygraph is a **polygraphic resolution** if each cellular extension X_{n+1} is **acyclic**.

Standard polygraphic resolution

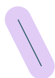
Every operad P is presented by a polygraphic resolution called the **standard polygraphic resolution** $\mathrm{Std}(P)$ of P:

- The indexed set $\mathrm{Std}(P)_0$ of generators is P, seen as an indexed set. Given an element $u \in P$, we denote the associated element $[u]$ in $\mathrm{Std}(P)_0$.
- The indexed set $\mathrm{Std}(P)_1$ of rewriting rules consists of a rule $([u] \mid [v_1] \cdots [v_k]) \to [(u \mid \vec{v})]$ for every $u, v_1, \ldots, v_k \in P$.
- For higher dimensions, we define $\mathrm{Std}(P)_{n+1}$ as the cellular extension containing a generating $(n+1)$-cell $f \to g$ for all pairs of parallel n-cells.

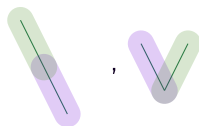
Overlapping polygraphic resolution

Let X be a convergent 1-polygraph. We construct the **overlapping polygraphic resolution** $\mathrm{Ov}(X)$ on X. The elements of $\mathrm{Ov}(X)_n$ correspond to certain overlappings of n rewriting rules:

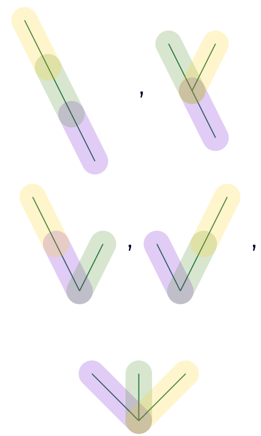
1-overlapping



2-overlapping



3-overlapping



and so on...

Overlapping polygraphic resolution

- As for tree monomials, we define a **path-lexicographic order** $\prec_{pl}$ on rules in context such that rewriting rules are smaller than generators.
- Define its overlappings $\mathrm{Ov}(X)$:
  - A 0-overlapping is an element of $\mathrm{Ov}(X)_0 := X_0$.
  - An n-overlapping $u_n \in \mathrm{Ov}(X)_n$ has n branches $C_1[\alpha_1] \prec_{pl} \cdots \prec_{pl} C_n[\alpha_n]$ with a common source $s_0(u_n)$. For a list of tree monomials $\vec{v}_{n+1}$, define the extensions on u_n in $\vec{v}_{n+1}$ as

$$E(u_n, \vec{v}_{n+1}) := \left\{ C[\alpha] \;\middle|\; \begin{array}{l} C[s_0(\alpha)] = (s_0(u_n) \mid \vec{v}_{n+1}), \\ C[\alpha] \succ_{pl} (C_n[\alpha_n] \mid \vec{v}_{n+1}) \end{array} \right\}$$

  An $(n+1)$-overlapping is a pair $(u_n, \vec{v}_{n+1})$ such that

$$\forall \vec{w}_{n+1} \subsetneq \vec{v}_{n+1}, \quad \max E(u_n, \vec{w}_{n+1}) \prec_{pl} E(u_n, \vec{v}_{n+1}).$$

  Its branches are $(C_1[\alpha_1] \mid \vec{v}_{n+1}) \prec_{pl} \cdots \prec_{pl} (C_n[\alpha_n] \mid \vec{v}_{n+1}) \prec_{pl} \max E(u_n, \vec{v}_{n+1})$ and its source is $(s_0(u_n) \mid \vec{v}_{n+1})$.
- The source and target maps are defined following a **right normalization strategy**.

Bimodule resolution from polygraphic resolution

Given a polygraphic resolution X presenting an operad P, we can construct an exact sequence of P-bimodules $P\langle X\rangle$:

$$\cdots \xrightarrow{\delta_n} P\langle X_n\rangle \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_1} P\langle X_1\rangle \xrightarrow{\delta_0} P\langle X_0\rangle$$

- For $n \geq 1$, $\delta_n(A)$ is given by $[s_n(A) - t_n(A)]$, replacing the various cellular compositions with addition.
- For $n = 0$, $\delta_0(\alpha) := [s_0(\alpha) - t_0(\alpha)]$, where

$$[(u \mid \vec{v})] := ([u] \mid \vec{\overline{v}}) + \sum_{i=1}^{k} (\overline{u} \mid \overline{v}_1 \cdots [v_i] \cdots \overline{v}_k).$$

Theorem [Malbos-R. 2020]

A operad P with a convergent quadratic polygraphic presentation X is Koszul.

Idea of proof.

- Extend the 1-polygraph X to the overlapping polygraphic resolution $\mathrm{Ov}(X)$.
- Study the induced P-bimodule resolution $(P\langle \mathrm{Ov}(X)_n \rangle)_n$, whose generators are concentrated on the superdiagonal.
- The Quillen homology of the operad P is then concentrated on the diagonal, and so P is Koszul.

And now...

We have defined the notion of **polygraphic resolution** of an operad.

- How to construct a resolution/cofibrant replacement in the category of **differential graded operads**?
- Does the overlapping resolution give a **minimal** cofibrant replacement?
- Can we generalize the overlapping polygraphic resolution to **quasi-terminating** presentations?
- Can shuffle operadic rewriting be generalized to **shuffle properads**?