Strategies and Resolutions

Algebraic Rewriting Seminar

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Introduction

We have seen that convergence implies decidability of the word problem. This leads us to ask:

Question

Does a finitely generated monoid with a decidable word problem always admit a finite convergent presentation?

- There are two approaches. Let $\mathcal{M}$ be a monoid ;
  - Homology : $\mathcal{M}$ has homological type FP_3 when there exists an exact sequence

$$P_3 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

    of modules over $\mathcal{M}$ with P_i is projective and finitely generated.
  - Homotopy : $\mathcal{M}$ is of finite derivation type FDT_3 if it admits a finite presentation with a finite homotopy basis.
- We have the following implications:

$\exists$ a finite convergent pres. $\qquad FDT_3 \qquad FP_3$

Goals

We want to extend these ideas to p-categories.

- Polygraphic resolutions :
  - Relate cofibrance in the model structure on $\infty\mathbf{Cat}$ to homotopy bases in every dimension, *i.e.* acyclicity.
- Homotopy :
  - Strategies : higher dimensional "base-points" via rewriting.
  - These are used to construct polygraphic resolutions, *i.e.* cofibrant replacements in the folk model structure of $\infty\mathbf{Cat}$.
  - Extends the finiteness condition FDT_3.
- Homology :
  - Abelianisation of homotopy produces a homological invariant extending FP_3.
  - Relate polygraphic resolutions to resolutions by modules.
- We obtain a rich interplay between computational and algebraico-topological properties.

Polygraphic resolutions

(n, p)-Polygraphs

We denote by n either a natural number or ∞.

- For $p \leq n$, an (n,p)-*category* is an n-category whose k-cells are invertible for every $k > p$.
- A model structure on (∞, p)-categories is inherited from ∞-categories via the adjunction:

- For $p \leq n$, an (n,p)-*polygraph* is data Σ consisting of:
  - a p-polygraph $(\Sigma_0, \ldots, \Sigma_p)$,
  - for every $p \leq k < n$, a cellular extension Σ_{k+1} of the free (k,p)-category

$$\Sigma_k^\top = \Sigma_p^*(\Sigma_{p+1})\cdots(\Sigma_k).$$

  The free (∞, p)-categories generated by such structures are cofibrant.

Polygraphic resolutions

Let $\mathcal{C}$ be a p-category.

- A *polygraphic resolution of* $\mathcal{C}$ is an acyclic (∞, p)-polygraph Σ such that $\overline{\Sigma} \cong \mathcal{C}$.
- If $p < n < \infty$, a *partial polygraphic resolution of length* n is an acyclic (n, p)-polygraph Σ such that $\overline{\Sigma}$ is isomorphic to $\mathcal{C}$.

Theorem

Let Σ be a polygraphic resolution of $\mathcal{C}$. The canonical projection $\Sigma^{\top} \twoheadrightarrow \mathcal{C}$ is a cofibrant approximation of $\mathcal{C}$.

$\mathcal{C}$ is said to be...

- of *finite ∞-derivation type* (FDT_{∞}) when it admits a finite polygraphic resolution,
- of *finite n-derivation type* (FDT_n) when it admits a finite partial polygraphic resolution of length n.

Illustration

Normalisation strategies and homotopy

Homotopy via rewriting

- In general, there are many choices of how to rewrite a given object; how do we know which path to take?
- For a convergent rewriting system, following any path from an object u, we arrive at the normal form $\hat{u}$.

- Strategies generalise this notion of normalisation to paths, paths between paths, ...

Homotopy via rewriting

- Strategies generalise this notion of normalisation to paths, paths between paths, ...
- These consist of
  - a notion of normal form in every dimension,
  - a reduction from every cell to its normal form.
- This information exhibits a homotopy from any k-cell to a "base point" in each hom-set.

Normalisation strategies

Let Σ be an (n, p)-polygraph.

- A *section of* Σ is the choice of a representative p-cell $\hat{u} : x \to y$ in $\Sigma^\top$ for every p-cell $u : x \to y$ of $\overline{\Sigma}$, such that

$$\widehat{1_x} \;=\; 1_x$$

  holds for every $(p-1)$-cell x of $\overline{\Sigma}$. Not functorial!
- This choice satisfies

$$\overline{u} = \overline{v} \text{ in } \overline{\Sigma} \qquad \Longleftrightarrow \qquad \hat{u} = \hat{v} \text{ in } \Sigma^*$$

Normalisation strategies

Let Σ be an (n,p)-polygraph.

- Choose such a (non-functorial) section $\widehat{(-)} : \overline{\Sigma} \rightarrowtail \Sigma_p^*$ of the canonical projection $\pi : \Sigma^\top \twoheadrightarrow \overline{\Sigma}$.
- Extend the section by induction via $\hat{f} = \sigma_{s(f)} \star_{k-1} \sigma_{t(f)}^-$ for a k-cell with $k > p$.

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- Extend the section by induction via $\hat{f} = \sigma_{s(f)} \star_{k-1} \sigma_{t(f)}^-$ for a k-cell with $k > p$.
- A *normalisation strategy for* Σ is a mapping σ of every k-cell f of $\Sigma^\top$, with $p \leq k < n$, to a $(k+1)$-cell

$$f \xrightarrow{\sigma_f} \hat{f}$$

  such that :
  - for every k-cell f, with $p \leq k < n$,

$$\sigma_{\hat{f}} = 1_{\hat{f}}$$

  - for every pair (f,g) of i-composable k-cells, with $p \leq i < k < n$,

$$\sigma_{f \star_i g} = \sigma_f \star_i \sigma_g \, .$$

Illustration: $(\infty, 1)$-polygraphs

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Lemma

Let Σ be an $(n,1)$-polygraph. Left (resp. right) normalisation strategies on Σ are in bijective correspondence with the families

$$\sigma_{\hat{u}\phi} : \hat{u}\phi \to \widehat{u\phi} \qquad (\textit{resp. } \sigma_{\phi\hat{u}} : \phi\hat{u} \to \widehat{\phi u})$$

of $(k+1)$-cells, indexed by k-cells ϕ of Σ, for $1 \leq k < n$, and by 1-cells u of $\overline{\Sigma}$ such that the composite k-cell $\overline{u}\phi$ (resp. $\phi\overline{u}$) exists.

Strategies and acyclicity

Theorem

Let Σ be an $(n,1)$-polygraph. We have that

$$\Sigma \text{ is acyclic} \quad \Longleftrightarrow \quad \Sigma \text{ admits a strategy.}$$

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Corollary

A 1*-category* $\mathcal{C}$ *is* FDT_n *if, and only if, there exists a finite* $(n, 1)$*-polygraph presenting* $\mathcal{C}$ *which admits a normalisation strategy.*

Strategies and acyclicity

Theorem

Let Σ be an $(n,1)$-polygraph. We have that

$$\Sigma \text{ is acyclic} \iff \Sigma \text{ admits a strategy.}$$

Corollary

A 1-category $\mathcal{C}$ is FDT_n if, and only if, there exists a finite $(n,1)$-polygraph presenting $\mathcal{C}$ which admits a normalisation strategy.

Now we want to relate convergence to the existence of a strategy…

Reduced polygraphs and ordering

- A 2-polygraph Σ is *reduced* when, for every 2-cell $\phi : u \Rightarrow v$ in Σ, the 1-cell u is a normal form for $\Sigma_2 \setminus \{\phi\}$ and v is a normal form for Σ_2.

Lemma

For every (finite) convergent n-polygraph, there exists a (finite) Tietze-equivalent, reduced and convergent n-polygraph.

Reduced polygraphs and ordering

- We define an ordering $\preceq$ on rewrite rules with the same source :
  - Let u a 1-cell of Σ^*.
  - For ϕ and ψ generating 2-cells of Σ, and rewriting steps

    $$f = v\phi v' \qquad \text{and} \qquad g = w\psi w',$$

    such that $s(f) = s(g) = u$ (a branching), we have

    $$f \preceq g \qquad \Longleftrightarrow \qquad v \text{ is a prefix of } w$$

  - This is a total order when Σ is reduced.

Canonical strategies: initialisation

Let Σ be a reduced 2-polygraph.

- Let u a 1-cell of Σ^* which is not a normal form.
- Since Σ is reduced, there are a finite number of steps with source u:

$$\lambda_u := f_1 \preceq f_2 \preceq \cdots \preceq f_{l-1} \preceq f_l =: \rho_u$$

- Denote respectively by λ_u and ρ_u the minimal and maximal rewriting steps.

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- Denote respectively by λ_u and ρ_u the minimal and maximal rewriting steps.
- Each results in a strategy defined by Noetherian induction.
- The *rightmost strategy* σ of Σ is defined by:
  - For a normal form $\hat{u}$, we (must) have

$$\sigma_{\hat{u}} = 1_{\hat{u}}.$$

  - On a reducible 1-cell u, set

$$\sigma_u = \rho_u \star_1 \sigma_{t(\lambda_u)}.$$

- Note that for every u, the 2-cell σ_u is an element of Σ^*.

Reduced critical branchings

Let Σ a reduced convergent 2-polygraph.

- What are the critical branchings of Σ?

- We conclude that a critical branching is of the form

$$(\phi\hat{v}, \rho_{u\hat{v}})$$

- When Σ is finite, there are a finite number of critical branchings.

Basis of generating confluences: induction step

- The *basis of generating confluences of* Σ is the cellular extension $c_2(\Sigma)$ of $\Sigma^\top$ made of one 3-cell

$$\begin{tikzcd}
 & \hat{u}\hat{v} \arrow[dr, Rightarrow, "\sigma_{\hat{u}\hat{v}}"] \arrow[d, phantom, "\Downarrow\omega_b"] & \\
u\hat{v} \arrow[ur, Rightarrow, "\phi\hat{v}"] \arrow[rr, Rightarrow, bend right, "\sigma_{u\hat{v}}"'] & {} & \hat{u}v
\end{tikzcd}$$

  for every critical branching b of Σ, where σ is the rightmost strategy.

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Lemma

The rightmost normalisation strategy of Σ *extends to a strategy of* $c_2(\Sigma)$.

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The rightmost normalisation strategy of Σ *extends to a strategy of* $c_2(\Sigma)$.

Proposition

The $(3,1)$*-polygraph* $c_2(\Sigma)$ *is acyclic.*

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Lemma

The rightmost normalisation strategy of Σ extends to a strategy of $c_2(\Sigma)$.

Corollary

A category with a finite convergent presentation is FDT_3.

In arbitrary dimensions

- An n*-fold branching of* Σ is a family $(f_1, \dots, f_n)$ of rewriting steps of Σ with the same source and such that $f_1 \preceq \cdots \preceq f_n$.
- We define local, aspherical, Peiffer, overlapping, minimal and critical n-fold branchings.
- An n-fold critical branching b of Σ must have shape

$$b = \left(c\hat{v},\ \rho_{u\hat{v}} \right)$$

  where c is a critical $(n-1)$-fold branching of Σ with source u.
- We again consider the cellular extension $c_n(\Sigma)$ of $c_{n-1}(\Sigma)^\top$ filling every critical branching.

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- We again consider the cellular extension $c_n(\Sigma)$ of $c_{n-1}(\Sigma)^\top$ filling every critical branching. We obtain the following:

Theorem

Every convergent 2-polygraph Σ extends to a Tietze-equivalent, acyclic $(\infty, 1)$-polygraph $c_\infty(\Sigma)$, whose generating n-cells, for every $n \geq 3$, are (indexed by) the critical $(n-1)$-fold branchings of Σ.

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Corollary

A category with a finite convergent presentation is FDT_∞.

Abelianisation and homology

Modules over monoids

Let $\mathcal{M}$ be a monoid.

- Recall that a *module* is an abelian group H equipped with an external action of $\mathcal{M}$:
  - For every $m \in \mathcal{M}$, a morphism

$$\begin{aligned} \tilde{m} : H &\longrightarrow H \\ h &\longmapsto m \cdot h, \end{aligned}$$

such that $\tilde{m} \circ \tilde{m}' = \widetilde{(m'm)}$.

Modules over monoids

Let $\mathcal{M}$ be a monoid.

- A *resolution* of $\mathcal{M}$ is an exact sequence of modules P_i

$$\cdots \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} P_1 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0.$$

  where $\mathbb{Z}$ is the trivial $\mathcal{M}$-module.
  A resolution is *partial of length* n when $\exists n$ s.t. $P_i = 0$ for $i > n$.

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  A resolution is *partial of length* n when $\exists n$ s.t. $P_i = 0$ for $i > n$.
- $\mathcal{M}$ is *of homological type* FP_n if there exists a resolution of $\mathbb{Z}$ by finitely generated projective modules.
- What does this mean in terms of presentations?
  - Generating elements of P_0 correspond to generators of $\mathcal{M}$.
  - Equations in P_0 are borders of free sums of generators of P_1. These generators correspond to relations.
  - Equations between relations are similarly "resolved" by elements of P_2, and so on…

Modules over monoids

Let $\mathcal{C}$ be a category.

- A *module* M over $\mathcal{C}$ is a functor

$$M : \mathcal{C} \longrightarrow \mathsf{Ab}.$$

- This is generalised by the notion of *natural system*:
  - The *factorisation category* $F\mathcal{C}$ of $\mathcal{C}$ consists of:
  - A natural system D over $\mathcal{C}$ is a functor

$$\begin{aligned} D : F\mathcal{C} &\longrightarrow \mathsf{Ab} \\ w &\longmapsto D_w \end{aligned}$$

Free natural system generated by Σ

Let $\mathcal{C}$ be a category and Σ an $(n,1)$-polygraph presenting $\mathcal{C}$.

- Given a family X of 1-cells of $\mathcal{C}$, we denote by $D_{\mathcal{C}}[X]$ the *free natural system on $\mathcal{C}$ generated by X*, which is defined by

$$D_{\mathcal{C}}[X] = \bigoplus_{u \in X} \mathbb{Z}F\mathcal{C}(u, -).$$

- We define free natural systems generated by Σ:
  - $D_{\mathcal{C}}[\Sigma_0]$ is generated by identities 1_u.

$$D_{\mathcal{C}}[\Sigma_0]_w = \langle \{(u,v) \mid uv = w\} \rangle.$$

  - For $1 \leq k < n$, $D_{\mathcal{C}}[\Sigma_k]$ is generated by a copy of $\overline{\phi}$ for every k-cell ϕ of Σ_k, *i.e.*

$$\begin{aligned} D_{\mathcal{C}}[\Sigma_k]_w &= \bigoplus_{\phi \in \Sigma_k} \mathbb{Z}F\mathcal{C}(\overline{\phi}, -) \\ &= \langle \{(u, \phi, v) \mid u\overline{\phi}v = w\} \rangle. \end{aligned}$$

- The generator $(u,\ \phi, v)$ is henceforth denoted by $u[\phi]v$.

FP_n and co. for categories

Let $\mathcal{C}$ be a category.

- Denote by $\mathbb{Z}$ the natural system on $\mathcal{C}$ which sends every arrow to $\mathbb{Z}$.
- We say that $\mathcal{C}$ is *of homological type* FP_n when the constant natural system $\mathbb{Z}$ is of type FP_n (viewed as a module over $F\mathcal{C}$).
- In other words, there exists a sequence of natural systems D_i such that for every 1-cell w of $\mathcal{C}$

  $$\cdots \xrightarrow{d_{n+1}} (D_n)_w \xrightarrow{d_n} (D_{n-1})_w \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} (D_0)_w \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0.$$

  is an exact sequence and $(D_i)_w$ is projective and finitely generated for all i.

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  is an exact sequence and $(D_i)_w$ is projective and finitely generated for all i.
- We also define homological invariants for right, left and bi-modules:

Description of free natural systems generated by Σ

Let Σ be a $(n,1)$-polygraph.

- Recall that we have a mapping

$$\begin{aligned}[-] : \Sigma_1 &\longrightarrow D_{\overline{\Sigma}}[\Sigma_1]_{\overline{x}} \\ x &\longmapsto [x] = (1_{s(x)}, x, 1_{t(x)}).\end{aligned}$$

- We extend this to Σ_1^* by induction on the size of $u \in \Sigma_1^*$:

$$[1_x] = 0 \qquad \text{and} \qquad [uv] = [u]\overline{v} + \overline{u}[v].$$

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- Similarly, for $1 < k \leq n$, a k-cell $f \in \Sigma_k^\top$ is associated to the element $[f]$ of $D_{\overline{\Sigma}}[\Sigma_k]_{\overline{f}}$ again by induction on its size:

$$[1_u] \;=\; 0, \qquad [f^-] \;=\; -[f], \qquad [f \star_i g] \;=\; \begin{cases} [f]\overline{g} + \overline{g}[f] & \text{if } i = 0 \\ [f] + [g] & \text{if not.} \end{cases}$$

- These extensions are well defined.

The Reidemeister-Fox-Squier complex (RFS)

Let Σ be a $(n,1)$-polygraph.

- For $1 \leq k \leq n$, we define the *k-th RFS boundary map of Σ*:

  $$\delta_k \ : \ D_{\overline{\Sigma}}[\Sigma_k] \longrightarrow D_{\overline{\Sigma}}[\Sigma_{k-1}]$$

  defined, on the generator $[\alpha]$:

  $$\delta_k[\alpha] \ = \begin{cases} (cl(\alpha), 1) - (1, cl(\alpha)) & \text{if } k = 1 \\ [s(\alpha)] - [t(\alpha)] & \text{if not.} \end{cases}$$

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- The *augmentation map ϵ of Σ* is defined, for every pair (u, v) of composable 1-cells of $\overline{\Sigma}$, by:

$$\epsilon(u, v) \ = \ 1.$$

The Reidemeister-Fox-Squier complex (RFS)

Let Σ be a $(n, 1)$-polygraph.

- By induction on the size of cells of $\Sigma^\top$, one proves that, for every k-cell f in $\Sigma^\top$, with $k \geq 1$, the following holds:

$$\delta_k[f] = \begin{cases} (cl(f), 1) - (1, cl(f)) & \text{if } k = 1 \\ [s(f)] - [t(f)] & \text{otherwise.} \end{cases}$$

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- As a consequence, for every $1 \leq k < n$, we have

$$\epsilon\delta_1 = 0 \qquad \text{and} \qquad \delta_k \delta_{k+1} = 0.$$

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$$\epsilon\delta_1 = 0 \qquad \text{and} \qquad \delta_k\delta_{k+1} = 0.$$

- The *Reidemeister-Fox-Squier (RFS) complex of* Σ is denoted by $D_{\overline{\Sigma}}[\Sigma]$:

$$D_{cl(\Sigma)}[\Sigma_n] \xrightarrow{\delta_n} D_{cl(\Sigma)}[\Sigma_{n-1}] \xrightarrow{\delta_{n-1}} \cdots$$

$$\cdots \xrightarrow{\delta_2} D_{cl(\Sigma)}[\Sigma_1] \xrightarrow{\delta_1} D_{cl(\Sigma)}[\Sigma_0] \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0.$$

Contracting homotopies from strategies

Fix a partial polygraphic resolution Σ of length $n \geq 1$ of a category $\mathcal{C}$.

- Since Σ is acyclic, it admits a left normalisation strategy σ.

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- We specify the corresponding morphisms of natural systems:

$$(\sigma_{-1})_w : \begin{array}{rcl} \mathbb{Z} & \to & D_{\mathcal{C}}[\Sigma_0]_w \\ 1 & \mapsto & (1, w) \end{array} \qquad (\sigma_0)_w : \begin{array}{rcl} D_{\mathcal{C}}[\Sigma_0]_w & \to & D_{\mathcal{C}}[\Sigma_1]_w \\ (u, v) & \mapsto & [\hat{u}]v \end{array}$$

$$(\sigma_k)_w : \begin{array}{rcl} D_{\mathcal{C}}[\Sigma_k]_w & \to & D_{\mathcal{C}}[\Sigma_{k+1}]_w \\ u[x]v & \mapsto & [\sigma_{\hat{u}x}]v \end{array}$$

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- We specify the corresponding morphisms of natural systems:

$$\begin{array}{rccc} (\sigma_{-1})_w : & \mathbb{Z} & \to & D_{\mathcal{C}}[\Sigma_0]_w \\ & 1 & \mapsto & (1, w) \end{array} \qquad \begin{array}{rccc} (\sigma_0)_w : & D_{\mathcal{C}}[\Sigma_0]_w & \to & D_{\mathcal{C}}[\Sigma_1]_w \\ & (u, v) & \mapsto & [\hat{u}]v \end{array}$$

$$\begin{array}{rccc} (\sigma_k)_w : & D_{\mathcal{C}}[\Sigma_k]_w & \to & D_{\mathcal{C}}[\Sigma_{k+1}]_w \\ & u[x]v & \mapsto & [\sigma_{\hat{u}x}]v \end{array}$$

Lemma

For every $k \in \{1, \ldots, n-1\}$, every k-cell f of $\Sigma^{\top}$ and every 1-cells u and v of $\mathcal{C}$ such that $ucl(f)v$ exists, we have:

$$\sigma_k(u[f]v) \;=\; [\sigma_{\hat{u}f}]v.$$

Convergence and homology

Theorem

If Σ is a (partial) polygraphic resolution (of length n) of a category $\mathcal{C}$, then the Reidemeister-Fox-Squier complex $D_{\mathcal{C}}[\Sigma]$ is a free (partial) resolution (of length n) of the constant natural system $\mathbb{Z}$ on $\mathcal{C}$.

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Corollary

The property FDT_n implies the property FP_n, for every $0 \leq n \leq \infty$.

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Corollary

If a category admits a finite and convergent presentation, then it is of homological type FP_∞.

Conclusion

Thank you