

Presenting isomorphic finitely presented modules by equivalent matrices: a constructive approach

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I. Motivations

- ▷ Equivalent linear functional systems
- ▷ Linear functional systems and module theory
- ▷ Fitting and Warfield's theorems

II. Module isomorphisms

- ▷ Module presentations and Malgrange's remark
- ▷ Morphisms of finitely presented modules
- ▷ Matrix characterisation of module isomorphisms

III. Constructive proofs of Fitting and Warfield's theorems

- ▷ Fitting's theorem
- ▷ Warfield's theorem

IV. Conclusion and perspectives

I. MOTIVATIONS

Linear functional systems

Matrix representation: consider the linear system $R\eta = 0$, where

→ $\eta = (\eta_1, \dots, \eta_p)^T \in \mathcal{F}^p$: unknown vector in the functional space \mathcal{F}

→ $R \in \mathbf{D}^{q \times p}$: matrix in the operator ring \mathbf{D} (\mathcal{F} is a left \mathbf{D} -module)

Example: consider the linear 3-order ODE

$$\ddot{\eta}(t) + a_2 \dot{\eta}(t) + a_1 \dot{\eta} + a_0 \eta(t) = 0, \quad a_i \in \mathbb{Q}$$

→ $\eta \in \mathcal{F} := C^\infty(I)$ and $R := (\partial^3 + a_2 \partial^2 + a_1 \partial + a_0) \in \mathbf{D} := \mathbb{Q}[\partial]$

Two notions of equivalent systems

Isomorphic solutions

Systems: $R\eta = 0 \quad R'\eta' = 0$

$$(R \in \mathbf{D}^{q \times p} \quad R' \in \mathbf{D}^{q' \times p'})$$

Equivalence: \exists an isomorphism

$$\ker_{\mathcal{F}}(R.) \simeq \ker_{\mathcal{F}}(R'.)$$

Equivalent matrices

Systems: $L\eta = 0 \quad L'\eta' = 0$

$$(L \in \mathbf{D}^{r \times c} \quad L' \in \mathbf{D}^{r' \times c'})$$

Equivalence: $\exists X, Y$ invertible s.t.

$$LX = YL'$$

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$$R := L$$

$$R' := L'$$

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$\ker_{\mathcal{F}}(R.) \simeq \ker_{\mathcal{F}}(R'.)$ does not imply:

R and R' are equivalent matrices

Counter-example: $\mathcal{F} := C^\infty(I)$, $\mathbf{D} := \mathbb{Q}[\partial]$, and

$$R\eta := (\partial^3 + a_2\partial^2 + a_1\eta + a_0)\eta = \ddot{\eta} + a_2\dot{\eta} + a_1\eta + a_0\eta = 0$$

$$R'\eta' := \begin{pmatrix} \partial & -1 & 0 \\ 0 & \partial & -1 \\ a_0 & a_1 & a_2 + \partial \end{pmatrix} \begin{pmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \dot{\eta}_0 - \eta_1 \\ \dot{\eta}_1 - \eta_2 \\ a_0\eta_0 + a_1\eta_1 + a_2\eta_2 + \dot{\eta}_2 \end{pmatrix} = 0$$

Then, $\ker_{\mathcal{F}}(R.) \simeq \ker_{\mathcal{F}}(R'.)$ is given by

$$\eta \mapsto (\eta \quad \dot{\eta} \quad \ddot{\eta})^T \quad \text{and} \quad \eta_0 \mapsto (\eta_0 \quad \eta_1 \quad \eta_2)^T$$

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A weaker result: R and R' may be extended into equivalent matrices

→ requires suitable properties from **module theory**

Module of a linear functional system

Consider the linear system $R\eta = 0$, where $\eta \in \mathcal{F}^p$ and $R \in \mathbf{D}^{q \times p}$

Construction: $\mathbf{M} := \text{coker}(\cdot R) = \mathbf{D}^{1 \times p} / \mathbf{D}^{1 \times q} R$

→ \mathbf{M} is presented by p generators y_1, \dots, y_p submitted to q relations

$$\forall 1 \leq i \leq q: \quad R_{i1} \cdot y_1 + \dots + R_{ip} \cdot y_p =_{\mathbf{M}} 0$$

Remark: \mathbf{M} is constructed such as the ring $\mathcal{O} := \mathbb{Q}[x_1, \dots, x_p] / I(f_1, \dots, f_q)$

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$$\mathbf{M} \simeq \mathbf{M}' \Rightarrow (\forall \mathcal{F}: \ker_{\mathcal{F}}(R.) \simeq \ker_{\mathcal{F}}(R'.))$$

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Objective: assuming $\mathbf{M} \simeq \mathbf{M}'$

→ extend R and R' into equivalent matrices

Theoretical results: due to Fitting and Warfield

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Theorems [Fitting 1936, Warfield 1978].

Let $R \in \mathbf{D}^{q \times p}$ and $R' \in \mathbf{D}^{q' \times p'}$ be such that $\mathbf{D}^{1 \times p} / \mathbf{D}^{1 \times q} R \simeq \mathbf{D}^{1 \times p'} / \mathbf{D}^{1 \times q'} R'$

Fitting's theorem: $L \in \mathbf{D}^{q+p'+(p+q) \times p+p'}$ and $L' \in \mathbf{D}^{(q+p')+(p+q') \times p+p'}$ are equivalent, where

$$L := \begin{pmatrix} R & 0 \\ 0 & \text{id}_{p'} \\ 0 & 0 \end{pmatrix} \leftarrow (p+q') \text{ rows} \qquad L' := \begin{pmatrix} 0 & 0 \\ \text{id}_p & 0 \\ 0 & R' \end{pmatrix} \leftarrow (q+p') \text{ rows}$$

Warfield's theorem: for $i, z \in \mathbb{N}$ small enough, $\tilde{L} \in \mathbf{D}^{q+(p'-i)+(p+q'-z) \times p+(p'-i)}$ and

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Objective of the talk

Construct invertible matrices X, Y, \tilde{X} , and \tilde{Y} s.t.

$$LX = YL' \qquad \text{and} \qquad \tilde{L}\tilde{X} = \tilde{Y}\tilde{L}'$$

II. MODULE ISOMORPHISMS

Module presentations/morphisms

Let $R \in \mathbf{D}^{q \times p}$ and $\mathbf{M} := \mathbf{D}^{1 \times p} / \mathbf{D}^{1 \times q} R$

Elements of \mathbf{M} : equivalence classes of rows $\lambda \in \mathbf{D}^{1 \times p}$ modulo the rows of R :

$$\forall 1 \leq i \leq q : \left(0 \dots \underbrace{1}_{i\text{-th}} \dots 0 \right) \begin{pmatrix} R_{11} & \dots & R_{1p} \\ \vdots & \dots & \vdots \\ R_{q1} & \dots & R_{qp} \end{pmatrix} = (R_{i1} \quad \dots \quad R_{ip})$$

→ \mathbf{M} has p generators $y_j := [(0 \dots 1 \dots 0)]$ and q relations

$$[(R_{i1} \quad \dots \quad R_{ip})] =_{\mathbf{M}} R_{i1} \cdot y_1 + \dots + R_{ip} \cdot y_p =_{\mathbf{M}} 0$$

Morphism $\mathbf{M} \xrightarrow{\varphi} \mathcal{F}$: uniquely determined by $(f_j := \varphi(y_j)) \in \mathcal{F}^p$ s.t.

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Malgrange's isomorphism:

$$\text{hom}_{\mathbf{D}}(\mathbf{M}, \mathcal{F}) \simeq \ker_{\mathcal{F}}(R.)$$

Example

System: consider the linear system of PDEs

$$\begin{cases} \partial_1 \xi_1 =_{\mathcal{F}} 0 \\ \frac{1}{2}(\partial_2 \xi_1 + \partial_1 \xi_2) =_{\mathcal{F}} 0 \\ \partial_2 \xi_2 =_{\mathcal{F}} 0 \end{cases} \iff R\xi := \begin{pmatrix} \partial_1 & 0 \\ \frac{1}{2}\partial_2 & \frac{1}{2}\partial_1 \\ 0 & \partial_2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} =_{\mathcal{F}} 0$$

Corresponding module: $\mathbf{D} := \mathbb{Q}[\partial_1, \partial_2]$ and $\mathbf{M} := \mathbf{D}^{1 \times 2} / \mathbf{D}^{1 \times 3} R$

→ \mathbf{M} has 2 generators y_1, y_2 submitted to 3 relations

$$\partial_1 \cdot y_1 =_{\mathbf{M}} 0, \quad \frac{1}{2} \partial_2 \cdot y_1 + \frac{1}{2} \partial_1 \cdot y_2 =_{\mathbf{M}} 0, \quad \partial_2 \cdot y_2 =_{\mathbf{M}} 0$$

Malgrange's iso. illustration: let $\varphi : \mathbf{M} \rightarrow \mathcal{F}$ and $\xi_j := \varphi(y_j)$. Then, $R\xi =_{\mathcal{F}} 0$:

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Finite presentations and exact sequences

Consider a morphism of (finitely presented left \mathbf{D} -)modules

$$\varphi: \mathbf{M} := \mathbf{D}^{1 \times p} / \mathbf{D}^{1 \times q} R \rightarrow \mathbf{M}' := \mathbf{D}^{1 \times p'} / \mathbf{D}^{1 \times q'} R'$$

Pullback of solutions: φ induces a morphism $\varphi^* : \ker_{\mathcal{F}}(R'.) \rightarrow \ker_{\mathcal{F}}(R.)$, corresponding to

$$\varphi^* : \text{hom}_{\mathbf{D}}(\mathbf{M}', \mathcal{F}) \rightarrow \text{hom}_{\mathbf{D}}(\mathbf{M}, \mathcal{F}), \quad \eta \mapsto \eta \circ \varphi$$

Question: what is the matrix representation of φ ? \rightarrow requires exact sequences

Exact sequences: the presentation of \mathbf{M} is equivalent to the following exact sequence

$$\mathbf{D}^{1 \times q} \xrightarrow{.R} \mathbf{D}^{1 \times p} \xrightarrow{\pi} \mathbf{M} \rightarrow 0$$

i.e., $(.R : \mathbf{D}^{1 \times q} \rightarrow \mathbf{D}^{1 \times p}, \mu \mapsto \mu R)$ and $(\pi : \mathbf{D}^{1 \times p} \rightarrow \mathbf{M}, \lambda \mapsto [\lambda])$ are s.t.

$$(\text{im}(.R) = \ker(\pi)) \quad \text{and} \quad (\text{im}(\pi) = \ker(0) = \mathbf{M}, \quad \text{i.e. } \pi \text{ is onto})$$

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Proposition. $(\exists \varphi : \mathbf{M} \rightarrow \mathbf{M}') \iff (\exists P \in \mathbf{D}^{p \times p'}, Q \in \mathbf{D}^{q \times q'} : RP = QR')$

In this case, the following exact and commutative diagram holds

$$\begin{array}{ccccccc}
 \mathbf{D}^{1 \times q} & \xrightarrow{\cdot R} & \mathbf{D}^{1 \times p} & \xrightarrow{\pi} & \mathbf{M} & \longrightarrow & 0 \\
 \cdot Q \downarrow & & \downarrow \cdot P & & \downarrow \varphi & & \\
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 \downarrow \cdot Q & & \downarrow \cdot P & & \downarrow \varphi & & \\
 \mathbf{D}^{1 \times q'} & \xrightarrow{\cdot R'} & \mathbf{D}^{1 \times p'} & \xrightarrow{\pi'} & \mathbf{M}' & \longrightarrow & 0
 \end{array}$$

Proof (sketch)

\implies : φ corresponds to $\mathbf{m}' \in \ker_{\mathbf{M}'}(R.) \subseteq \mathbf{M}'^p$. Let $P \in \mathbf{D}^{p \times p'}$ defined by

$$\forall 1 \leq i \leq p : \quad \mathbf{m}'_i = P_{i1} \cdot y'_1 + \cdots + P_{ip'} \cdot y'_{p'}$$

$\mathbf{m}' \in \ker_{\mathbf{M}'}(R.)$ means: the rows of RP are combinations of rows of R' , i.e.

$$\exists Q \in \mathbf{D}^{q \times q'} : \quad RP = QR'$$

Proposition. $(\exists \varphi : \mathbf{M} \rightarrow \mathbf{M}') \iff (\exists P \in \mathbf{D}^{p \times p'}, Q \in \mathbf{D}^{q \times q'} : RP = QR')$

In this case, the following exact and commutative diagram holds

$$\begin{array}{ccccccc}
 \mathbf{D}^{1 \times q} & \xrightarrow{\cdot R} & \mathbf{D}^{1 \times p} & \xrightarrow{\pi} & \mathbf{M} & \longrightarrow & 0 \\
 \cdot Q \downarrow & & \downarrow \cdot P & & \downarrow \varphi & & \\
 \mathbf{D}^{1 \times q'} & \xrightarrow{\cdot R'} & \mathbf{D}^{1 \times p'} & \xrightarrow{\pi'} & \mathbf{M}' & \longrightarrow & 0
 \end{array}$$

Proof (sketch)

\implies : φ corresponds to $\mathbf{m}' \in \ker_{\mathbf{M}'}(R) \subseteq \mathbf{M}'^p$. Let $P \in \mathbf{D}^{p \times p'}$ defined by

$$\forall 1 \leq i \leq p: \quad \mathbf{m}'_i = P_{i1} \cdot y'_1 + \cdots + P_{ip'} \cdot y'_{p'}$$

$\mathbf{m}' \in \ker_{\mathbf{M}'}(R)$ means: the rows of RP are combinations of rows of R' , i.e.

$$\exists Q \in \mathbf{D}^{q \times q'} : \quad RP = QR'$$

\impliedby : let $\varphi([\lambda]) := [\lambda P]$; φ is well-defined iff $([\lambda] =_{\mathbf{M}} 0 \Rightarrow [\lambda P] =_{\mathbf{M}'} 0)$

By exactness: $[\lambda] =_{\mathbf{M}} 0 \Rightarrow (\exists \mu \in \mathbf{D}^{1 \times q} : \lambda = \mu R)$. Hence, we get:

$$[\lambda P] =_{\mathbf{M}'} [\mu RP] =_{\mathbf{M}'} [\mu QR'] =_{\mathbf{M}'} 0$$

Example

Systems: consider the following two systems of PDEs

$$R\xi := \begin{pmatrix} \partial_1 & 0 \\ \frac{1}{2}\partial_2 & \frac{1}{2}\partial_1 \\ 0 & \partial_2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} =_{\mathcal{F}} 0 \quad R'\zeta := \begin{pmatrix} \partial_1 & 0 & 0 \\ \partial_2 & -1 & 0 \\ 0 & \partial_1 & 0 \\ 0 & 1 & \partial_1 \\ 0 & 0 & \partial_2 \\ 0 & \partial_2 & 0 \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix} =_{\mathcal{F}} 0$$

Morphism: let $\varphi : \mathbf{M} \rightarrow \mathbf{M}'$ induced by

$$P := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad Q := \frac{1}{2} \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

The following equations hold

$$RP = \begin{pmatrix} \partial_1 & 0 & 0 \\ \frac{1}{2}\partial_2 & 0 & \frac{1}{2}\partial_1 \\ 0 & 0 & \partial_2 \end{pmatrix} = QR'$$

Pullback of solutions:

$$\varphi^* : \ker_{\mathcal{F}}(R'.) \rightarrow \ker_{\mathcal{F}}(R.), \quad \zeta := (\zeta_1 \quad \zeta_2 \quad \zeta_3)^T \mapsto \xi := P\zeta = (\zeta_1 \quad \zeta_3)^T$$

Theorem [Cluzeau - Quadrat 2011].

Let $\varphi : \mathbf{M} \rightarrow \mathbf{M}'$ be a morphism corresponding to matrices P, Q s.t. $RP = QR'$.

Then, φ is an isomorphism iff $\exists P' \in \mathbf{D}^{p' \times p}$, $Q' \in \mathbf{D}^{q' \times q}$, $Z \in \mathbf{D}^{p \times q}$, $Z' \in \mathbf{D}^{p' \times q'}$:

$$\begin{array}{ccccccc}
 \mathbf{D}^{1 \times q} & \xrightarrow{\cdot R} & \mathbf{D}^{1 \times p} & \xrightarrow{\pi} & \mathbf{M} & \longrightarrow & 0 \\
 \cdot Q \downarrow & & \downarrow \cdot P & & \downarrow \varphi & & \\
 \mathbf{D}^{1 \times q'} & \xrightarrow{\cdot R'} & \mathbf{D}^{1 \times p'} & \xrightarrow{\pi'} & \mathbf{M}' & \longrightarrow & 0
 \end{array}$$

s.t. the following relations hold

$$R'P' = Q'R$$

$$PP' + ZR = \text{id}_p$$

$$P'P + Z'R' = \text{id}_{p'}$$

Theorem [Cluzeau - Quadrat 2011].

Let $\varphi : \mathbf{M} \rightarrow \mathbf{M}'$ be a morphism corresponding to matrices P, Q s.t. $RP = QR'$.

Then, φ is an isomorphism iff $\exists P' \in \mathbf{D}^{p' \times p}, Q' \in \mathbf{D}^{q' \times q}, Z \in \mathbf{D}^{p \times q}, Z' \in \mathbf{D}^{p' \times q'}$:

$$\begin{array}{ccccccc}
 \mathbf{D}^{1 \times q} & \xrightarrow{\cdot R} & \mathbf{D}^{1 \times p} & \xrightarrow{\pi} & \mathbf{M} & \longrightarrow & 0 \\
 \downarrow \cdot Q & \uparrow \cdot Q' & \downarrow \cdot P & & \downarrow \varphi & & \\
 \mathbf{D}^{1 \times q'} & \xleftarrow{\cdot Z'} & \mathbf{D}^{1 \times p'} & \xrightarrow{\pi'} & \mathbf{M}' & \longrightarrow & 0 \\
 & \xrightarrow{\cdot R'} & & & & &
 \end{array}$$

$\xleftarrow{\cdot Z}$ (between $\mathbf{D}^{1 \times q}$ and $\mathbf{D}^{1 \times p}$)
 $\xleftarrow{\cdot P'}$ (between $\mathbf{D}^{1 \times p}$ and $\mathbf{D}^{1 \times p'}$)

s.t. the following relations hold

$$R'P' = Q'R$$

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Theorem [Cluzeau - Quadrat 2011].

Let $\varphi : \mathbf{M} \rightarrow \mathbf{M}'$ be a morphism corresponding to matrices P, Q s.t. $RP = QR'$.

Then, φ is an isomorphism iff $\exists P' \in \mathbf{D}^{p' \times p}$, $Q' \in \mathbf{D}^{q' \times q}$, $Z \in \mathbf{D}^{p \times q}$, $Z' \in \mathbf{D}^{p' \times q'}$:

$$\begin{array}{ccccccc}
 \mathbf{D}^{1 \times r} & \xrightleftharpoons[\cdot Z_2]{\cdot R_2} & \mathbf{D}^{1 \times q} & \xrightleftharpoons[\cdot Z]{\cdot R} & \mathbf{D}^{1 \times p} & \xrightarrow{\pi} & \mathbf{M} \longrightarrow 0 \\
 & & \downarrow \cdot Q \quad \uparrow \cdot Q' & & \downarrow \cdot P \quad \uparrow \cdot P' & & \downarrow \varphi \\
 \mathbf{D}^{1 \times r'} & \xrightleftharpoons[\cdot R'_2]{\cdot Z'_2} & \mathbf{D}^{1 \times q'} & \xrightleftharpoons[\cdot R']{\cdot Z'} & \mathbf{D}^{1 \times p'} & \xrightarrow{\pi'} & \mathbf{M}' \longrightarrow 0
 \end{array}$$

s.t. the following relations hold

$$R'P' = Q'R$$

$$PP' + ZR = \text{id}_p$$

$$P'P + Z'R' = \text{id}_{p'}$$

Moreover, there exist $R_2 \in \mathbf{D}^{r \times q}$, $Z_2 \in \mathbf{D}^{q \times r}$, $R'_2 \in \mathbf{D}^{r' \times q'}$, $Z'_2 \in \mathbf{D}^{q' \times r'}$ s.t.

$$\ker(\cdot R) = \mathbf{D}^{1 \times r} R_2$$

$$QQ' + RZ + Z_2R_2 = \text{id}_q$$

$$\ker(\cdot R') = \mathbf{D}^{1 \times r'} R'_2$$

$$Q'Q + R'Z' + Z'_2R'_2 = \text{id}_{q'}$$

Main steps of the proof (suppose φ is an isomorphism)

Objective: construct an exact diagram and check **various relations**

$$\begin{array}{ccccccccc}
 \mathbf{D}^{1 \times r} & \xrightarrow{\cdot R_2} & \mathbf{D}^{1 \times q} & \xrightarrow{\cdot R} & \mathbf{D}^{1 \times p} & \xrightarrow{\pi} & \mathbf{M} & \longrightarrow & 0 \\
 & \xleftarrow{\cdot Z_2} & & \xleftarrow{\cdot Z} & & & \downarrow \varphi & & \\
 & & \cdot Q \downarrow & & \cdot P' \downarrow & & & & \\
 & & \uparrow \cdot Q' & & \uparrow \cdot P & & & & \\
 \mathbf{D}^{1 \times r'} & \xleftarrow{\cdot Z'_2} & \mathbf{D}^{1 \times q'} & \xleftarrow{\cdot Z'} & \mathbf{D}^{1 \times p'} & \xrightarrow{\pi'} & \mathbf{M}' & \longrightarrow & 0 \\
 & \xrightarrow{\cdot R'_2} & & \xrightarrow{\cdot R'} & & & & &
 \end{array}$$

Main steps of the proof (suppose φ is an isomorphism)

Objective: construct an exact diagram and check **various relations**

$$\begin{array}{ccccccccc}
 \mathbf{D}^{1 \times r} & \xrightarrow{\cdot R_2} & \mathbf{D}^{1 \times q} & \xrightarrow{\cdot R} & \mathbf{D}^{1 \times p} & \xrightarrow{\pi} & \mathbf{M} & \longrightarrow & 0 \\
 & \xleftarrow{\cdot Z_2} & & \xleftarrow{\cdot Z} & & & \downarrow \varphi & & \\
 & & \cdot Q \downarrow & & \cdot P' \downarrow & & & & \\
 & & \uparrow \cdot Q' & & \uparrow \cdot P & & & & \\
 \mathbf{D}^{1 \times r'} & \xleftarrow{\cdot Z'_2} & \mathbf{D}^{1 \times q'} & \xleftarrow{\cdot Z'} & \mathbf{D}^{1 \times p'} & \xrightarrow{\pi'} & \mathbf{M}' & \longrightarrow & 0 \\
 & \xrightarrow{\cdot R'_2} & & \xrightarrow{\cdot R'} & & & & &
 \end{array}$$

$R'P' = Q'R$: due to the existence of φ^{-1}

Main steps of the proof (suppose φ is an isomorphism)

Objective: construct an exact diagram and check **various relations**

$$\begin{array}{ccccccccc}
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 & \xleftarrow{\cdot Z_2} & & \xleftarrow{\cdot Z} & & & \downarrow \varphi & & \\
 & & \cdot Q \downarrow & & \cdot P' \downarrow & & & & \\
 & & \cdot Q' \uparrow & & \cdot P \downarrow & & & & \\
 \mathbf{D}^{1 \times r'} & \xleftarrow{\cdot Z'_2} & \mathbf{D}^{1 \times q'} & \xleftarrow{\cdot Z'} & \mathbf{D}^{1 \times p'} & \xrightarrow{\pi'} & \mathbf{M}' & \longrightarrow & 0 \\
 & \xrightarrow{\cdot R'_2} & & \xrightarrow{\cdot R'} & & & & &
 \end{array}$$

$R'P' = Q'R$: due to the existence of φ^{-1}

$PP' + ZR = \text{id}_p$: (the proof of $P'P + Z'R' = \text{id}_{p'}$ is analogous)

$$\begin{aligned}
 \left(\forall \lambda \in \mathbf{D}^{1 \times p} : [\lambda] = \varphi^{-1} \circ \varphi([\lambda]) = [\lambda PP'] \right) &\Rightarrow \text{im} \left(\cdot (\text{id}_p - PP') \right) \subseteq \mathbf{D}^{1 \times q} R \\
 &\Rightarrow \exists Z \in \mathbf{D}^{p \times q} : \text{id}_p - PP' = ZR
 \end{aligned}$$

Main steps of the proof (suppose φ is an isomorphism)

Objective: construct an exact diagram and check **various relations**

$$\begin{array}{ccccccccc}
 \mathbf{D}^{1 \times r} & \xrightarrow{\cdot R_2} & \mathbf{D}^{1 \times q} & \xrightarrow{\cdot R} & \mathbf{D}^{1 \times p} & \xrightarrow{\pi} & \mathbf{M} & \longrightarrow & 0 \\
 & \xleftarrow{\cdot Z_2} & & \xleftarrow{\cdot Z} & & & \downarrow \varphi & & \\
 & & \cdot Q \downarrow & \uparrow \cdot Q' & \cdot P' \downarrow & \uparrow \cdot P & & & \\
 \mathbf{D}^{1 \times r'} & \xleftarrow{\cdot Z'_2} & \mathbf{D}^{1 \times q'} & \xleftarrow{\cdot Z'} & \mathbf{D}^{1 \times p'} & \xrightarrow{\pi'} & \mathbf{M}' & \longrightarrow & 0 \\
 & \xrightarrow{\cdot R'_2} & & \xrightarrow{\cdot R'} & & & & &
 \end{array}$$

$R'P' = Q'R'$: due to the existence of φ^{-1}

$PP' + ZR = \text{id}_p$: (the proof of $P'P + Z'R' = \text{id}_{p'}$ is analogous)

$$\begin{aligned}
 \left(\forall \lambda \in \mathbf{D}^{1 \times p} : [\lambda] = \varphi^{-1} \circ \varphi([\lambda]) = [\lambda PP'] \right) &\Rightarrow \text{im} \left(\cdot (\text{id}_p - PP') \right) \subseteq \mathbf{D}^{1 \times q} R \\
 &\Rightarrow \exists Z \in \mathbf{D}^{p \times q} : \text{id}_p - PP' = ZR
 \end{aligned}$$

$QQ' + RZ + Z_2R_2 = \text{id}_q$ and $Q'Q + R'Z' + Z'_2R'_2 = \text{id}_{q'}$: basic homological algebra

Example

Morphism: $\varphi : \mathbf{M} := \mathbf{D}^{1 \times 2} / \mathbf{D}^{1 \times 3} R \rightarrow \mathbf{M}' := \mathbf{D}^{1 \times 3} / \mathbf{D}^{1 \times 6} R'$ defined by

$$R := \begin{pmatrix} \partial_1 & 0 \\ \frac{1}{2}\partial_2 & \frac{1}{2}\partial_1 \\ 0 & \partial_2 \end{pmatrix} \quad R' := \begin{pmatrix} \partial_1 & 0 & 0 \\ \partial_2 & -1 & 0 \\ 0 & \partial_1 & 0 \\ 0 & 1 & \partial_1 \\ 0 & 0 & \partial_2 \\ 0 & \partial_2 & 0 \end{pmatrix} \quad P := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$Q := \frac{1}{2} \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

Fact: φ is an isomorphism, where φ^{-1} is defined by

$$P' := \begin{pmatrix} 1 & 0 \\ \partial_2 & 0 \\ 0 & 1 \end{pmatrix} \quad Q' := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ \partial_2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 2\partial_2 & -\partial_1 \end{pmatrix} \rightsquigarrow R'P' = \begin{pmatrix} \partial_1 & 0 \\ 0 & 0 \\ \partial_1\partial_2 & 0 \\ \partial_2 & \partial_1 \\ 0 & \partial_2 \\ \partial_2^2 & 0 \end{pmatrix} = Q'R$$

Indeed: $\forall \lambda \in \mathbf{D}^{1 \times 2} : \varphi^{-1} \circ \varphi([\lambda]) = [\lambda P P'] = [\lambda \text{id}_2] = [\lambda]$ and

$$\forall \lambda' \in \mathbf{D}^{1 \times 3} : \varphi \circ \varphi^{-1}([\lambda']) = [\lambda' P' P] = \left[\lambda' \begin{pmatrix} 1 & 0 & 0 \\ \partial_2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] = [\lambda'_1 + \lambda'_2 \partial_2 \quad 0 \quad \lambda'_3] = [\lambda']$$

III. CONSTRUCTIVE PROOFS OF FITTING AND WARFIELD'S THEOREMS

III.1. FITTING'S THEOREM

Preliminaries

$$\text{Fix } (\mathbf{M} := \mathbf{D}^{1 \times p} / \mathbf{D}^{1 \times q} R \quad \mathbf{M}' := \mathbf{D}^{1 \times p'} / \mathbf{D}^{1 \times q'} R') \leftrightarrow (R\eta =_{\mathcal{F}} 0 \quad R'\eta' =_{\mathcal{F}} 0)$$

Lemma: $\mathbf{M} \simeq \mathbf{D}^{1 \times c} / \mathbf{D}^{1 \times r} L$ and $\mathbf{M}' \simeq \mathbf{D}^{1 \times c} / \mathbf{D}^{1 \times r} L'$, where

$$\begin{cases} c := p + p' \\ r := q + p' + p + q' \end{cases} \quad L := \begin{pmatrix} R & 0 \\ 0 & \text{id}_{p'} \\ 0 & 0 \end{pmatrix} \leftarrow (p + q') \quad L' := \begin{pmatrix} 0 & 0 \\ \text{id}_p & 0 \\ 0 & R' \end{pmatrix} \leftarrow (q + p')$$

Extended systems: L and L' correspond to the systems

$$L \begin{pmatrix} \eta \\ \xi \end{pmatrix} =_{\mathcal{F}} \begin{pmatrix} R\eta \\ \xi \\ 0 \end{pmatrix} =_{\mathcal{F}} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad L' \begin{pmatrix} \eta' \\ \xi' \end{pmatrix} =_{\mathcal{F}} \begin{pmatrix} R'\eta' \\ \xi' \\ 0 \end{pmatrix} =_{\mathcal{F}} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Preliminaries

Fix $(\mathbf{M} := \mathbf{D}^{1 \times p} / \mathbf{D}^{1 \times q} R \quad \mathbf{M}' := \mathbf{D}^{1 \times p'} / \mathbf{D}^{1 \times q'} R')$ \leftrightarrow $(R\eta =_{\mathcal{F}} 0 \quad R'\eta' =_{\mathcal{F}} 0)$

Lemma: $\mathbf{M} \simeq \mathbf{D}^{1 \times c} / \mathbf{D}^{1 \times r} L$ and $\mathbf{M}' \simeq \mathbf{D}^{1 \times c} / \mathbf{D}^{1 \times r} L'$, where

$$\begin{cases} c := p + p' \\ r := q + p' + p + q' \end{cases} \quad L := \begin{pmatrix} R & 0 \\ 0 & \text{id}_{p'} \\ 0 & 0 \end{pmatrix} \leftarrow (p + q') \quad L' := \begin{pmatrix} 0 & 0 \\ \text{id}_p & 0 \\ 0 & R' \end{pmatrix} \leftarrow (q + p')$$

Extended systems: L and L' correspond to the systems

$$L \begin{pmatrix} \eta \\ \xi \end{pmatrix} =_{\mathcal{F}} \begin{pmatrix} R\eta \\ \xi \\ 0 \end{pmatrix} =_{\mathcal{F}} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad L' \begin{pmatrix} \eta' \\ \xi' \end{pmatrix} =_{\mathcal{F}} \begin{pmatrix} R'\eta' \\ \xi' \\ 0 \end{pmatrix} =_{\mathcal{F}} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Fitting's theorem: if $\mathbf{M} \simeq \mathbf{M}'$, then L and L' are equivalent

Example

Isomorphic modules: correspond to the matrices

$$R := \begin{pmatrix} \partial_1 & 0 \\ \frac{1}{2}\partial_2 & \frac{1}{2}\partial_1 \\ 0 & \partial_2 \end{pmatrix}$$

$$R' := \begin{pmatrix} \partial_1 & 0 & 0 \\ \partial_2 & -1 & 0 \\ 0 & \partial_1 & 0 \\ 0 & 1 & \partial_1 \\ 0 & 0 & \partial_2 \\ 0 & \partial_2 & 0 \end{pmatrix}$$

From Fitting's theorem: $L, L' \in \mathbf{D}^{5 \times 14}$ are equivalent matrices

$$L := \begin{pmatrix} \partial_1 & 0 & 0 & 0 & 0 \\ \frac{1}{2}\partial_2 & \frac{1}{2}\partial_1 & 0 & 0 & 0 \\ 0 & \partial_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$L' := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \partial_1 & 0 & 0 \\ 0 & 0 & \partial_2 & -1 & 0 \\ 0 & 0 & 0 & \partial_1 & 0 \\ 0 & 0 & 0 & 1 & \partial_1 \\ 0 & 0 & 0 & 0 & \partial_2 \\ 0 & 0 & 0 & \partial_2 & 0 \end{pmatrix}$$

Theorem [Cluzeau - Quadrat 2011].

Let $\varphi : \mathbf{M} := \mathbf{D}^{1 \times p} / \mathbf{D}^{1 \times q} R \xrightarrow{\sim} \mathbf{M}' := \mathbf{D}^{1 \times p'} / \mathbf{D}^{1 \times q'} R'$ corresponding to

$$\begin{array}{ccccccc}
 \mathbf{D}^{1 \times r} & \xrightarrow{\cdot R_2} & \mathbf{D}^{1 \times q} & \xrightarrow{\cdot R} & \mathbf{D}^{1 \times p} & \xrightarrow{\pi} & \mathbf{M} \longrightarrow 0 \\
 & \xleftarrow{\cdot Z_2} & & \xleftarrow{\cdot Z} & & & \downarrow \varphi \\
 & & \cdot Q \downarrow & \uparrow \cdot Q' & \cdot P' \uparrow \downarrow \cdot P & & \\
 \mathbf{D}^{1 \times r'} & \xleftarrow{\cdot Z'_2} & \mathbf{D}^{1 \times q'} & \xleftarrow{\cdot Z'} & \mathbf{D}^{1 \times p'} & \xrightarrow{\pi'} & \mathbf{M}' \longrightarrow 0 \\
 & \xrightarrow{\cdot R'_2} & & \xrightarrow{\cdot R'} & & &
 \end{array}$$

Then, the pairs of invertible matrices $(X, X^{-}) \in \mathbf{D}^{c \times c}$ and $(Y, Y^{-}) \in \mathbf{D}^{r \times r}$

$$X := \begin{pmatrix} \text{id}_p & P \\ -P' & \text{id}_{p'} - P'P \end{pmatrix} \quad X^{-} := \begin{pmatrix} \text{id}_p & -PP' \\ P' & \text{id}_{p'} \end{pmatrix}$$

$$Y := \begin{pmatrix} \text{id}_q & 0 & R & Q \\ 0 & \text{id}_{p'} & -P' & Z' \\ -Z & P & 0 & PZ' - ZQ \\ -Q' & -R' & 0 & Z'_2 R'_2 \end{pmatrix} \quad Y^{-} := \begin{pmatrix} Z_2 R_2 & 0 & -R & -Q \\ P'Z - Z'Q' & 0 & P' & -Z' \\ Z & -P & \text{id}_p & 0 \\ Q' & R' & 0 & \text{id}_{q'} \end{pmatrix}$$

are such that

$$LX := \begin{pmatrix} R & 0 \\ 0 & \text{id}_{p'} \\ 0 & 0 \end{pmatrix} X = Y \begin{pmatrix} 0 & 0 \\ \text{id}_p & 0 \\ 0 & R' \end{pmatrix} =: YL'$$

Comments

Ideas of the proof: compute matrix products using relations between P 's, Q 's, R 's, Z 's

Improvement: if R and R' have full row rank, we may remove the 0 lines of L and L'

Practical use:

- start with the two modules presented by R and R'
- if an isomorphism exists, compute P , P' , Q , and Q'
- compute Z , Z' , R_2 , R'_2 , Z_2 , and Z'_2
- deduce the expressions of X , X^- , Y , and Y^-

Implementation: package `OreMorphisms` (*Maple*) when \mathbf{D} is an Ore algebra

- ODE/PDEs operators, shift operators, time-delay operators, difference operators

Comments

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Implementation: package `OreMorphisms` (*Maple*) when \mathbf{D} is an **Ore algebra**

- ODE/PDEs operators, shift operators, time-delay operators, difference operators

Running example

$$X = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 \\ -\partial_2 & 0 & -\partial_2 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$

$$X^- = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 & 0 \\ \partial_2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$Y = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \partial_1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{2}\partial_2 & \frac{1}{2}\partial_1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \partial_2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\partial_2 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -\partial_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\partial_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\partial_2 & 0 & 0 & 0 & -\partial_1 & 0 & 0 & 0 & -\partial_2 & \partial_1 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & -1 & -\partial_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -\partial_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2\partial_2 & \partial_1 & 0 & -\partial_2 & 0 & 0 & 0 & 0 & 0 & 0 & -\partial_2 & \partial_1 & 1 \end{pmatrix}$$

Running example (end)

$$Y^- = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -\partial_1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\partial_2 & -\frac{1}{2}\partial_1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\partial_2 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\partial_1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \partial_1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & \partial_2 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \partial_2 & -2\partial_1 & 0 & 0 & \partial_1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \partial_1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \partial_2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\partial_1 & 0 & \partial_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

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Summary: $X, X^- \in \mathbf{D}^{5 \times 5}$ and $Y, Y^- \in \mathbf{D}^{14 \times 14}$

III.2. WARFIELD'S THEOREM

Next objective

if $\mathbf{M} \simeq \mathbf{M}'$: reduce the size of L and L'

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$$\begin{cases} z \leq \min(p + q', q + p') \\ \text{sr}(\mathbf{D}) \leq \max(p + q' - z, q + p' - z) \\ i \leq \min(p, p') \\ \text{sr}(\mathbf{D}) \leq \max(p - i, p' - i) \end{cases}$$

where $\text{sr}(\mathbf{D})$ is the stable rank of \mathbf{D} . If $\mathbf{M} \simeq \mathbf{M}'$, then the following matrices are equivalent

$$\tilde{L} := \begin{pmatrix} R & 0 \\ 0 & \text{id}_{p'-i} \\ 0 & 0 \end{pmatrix} \leftarrow (p + q' - z)$$

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where $\text{sr}(\mathbf{D})$ is the **stable rank** of \mathbf{D} . If $\mathbf{M} \simeq \mathbf{M}'$, then the following matrices are equivalent

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Stable rank

Preliminaries: let $u := (u_1 \ \dots \ u_n)^T \in \mathbf{D}^n$ be a column vector

▷ u is called unimodular if there exists a row vector $v \in \mathbf{D}^{1 \times n}$ s.t. $vu = 1$

▷ u is called stable if there exists $d_1, \dots, d_{n-1} \in \mathbf{D}$ s.t.

$$(u_1 + d_1 u_n \ \dots \ u_{n-1} + d_{n-1} u_n)^T \text{ is unimodular}$$

Definition: the stable rank of \mathbf{D} is the smallest integer $\text{sr}(\mathbf{D})$ s.t.

$$\forall n > \text{sr}(\mathbf{D}) : \quad u \in \mathbf{D}^n \text{ is unimodular} \quad \Rightarrow \quad u \text{ is stable}$$

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Examples:

- ▷ If \mathbf{D} is a principal domain, then $\text{sr}(\mathbf{D}) \geq 2$
- ▷ For every $n \geq 1$, $\text{sr}(\mathbb{Q}[x_1, \dots, x_n]) = n + 1$
- ▷ Stafford's theorem: if $\text{char}(\mathbb{K}) = 0$, then $\text{sr}(A_n(\mathbb{K})) = \text{sr}(B_n(\mathbb{K})) = 2$
 $(A_n(\mathbb{K})/B_n(\mathbb{K}) := \text{polynomial/rational Weyl algebra})$

Removing one zero line

From Fitting's theorem: there is a commutative diagram

$$\begin{array}{ccc} \mathbf{D}^{1 \times r} & \xrightarrow{\cdot Y} & \mathbf{D}^{1 \times r} \\ \cdot L \downarrow & & \downarrow \cdot L' \\ \mathbf{D}^{1 \times c} & \xrightarrow{\cdot X} & \mathbf{D}^{1 \times c} \end{array}$$

Objective: prove that L_1 and L'_1 are equivalent

$$L_1 := \begin{pmatrix} R & 0 \\ 0 & \text{id}_{p'} \\ 0 & 0 \end{pmatrix} \leftarrow (p + q' - 1) \qquad L'_1 := \begin{pmatrix} 0 & 0 \\ \text{id}_p & 0 \\ 0 & R' \end{pmatrix} \leftarrow (q + p' - 1)$$

Method: construct a commutative diagram s.t. the top row is invertible

$$\begin{array}{ccccccccc} \mathbf{D}^{1 \times (r-1)} & \xrightarrow{\cdot H_1} & \mathbf{D}^{1 \times r} & \xrightarrow{\cdot W_1} & \mathbf{D}^{1 \times r} & \xrightarrow{\cdot Y} & \mathbf{D}^{1 \times r} & \xrightarrow{\cdot G_1} & \mathbf{D}^{1 \times (r-1)} \\ \cdot L_1 \downarrow & & \downarrow \cdot L & & \downarrow \cdot L & & \downarrow L' & & \downarrow \cdot L'_1 \\ \mathbf{D}^{1 \times c} & \xlongequal{\quad} & \mathbf{D}^{1 \times c} & \xlongequal{\quad} & \mathbf{D}^{1 \times c} & \xrightarrow{\cdot X} & \mathbf{D}^{1 \times c} & \xlongequal{\quad} & \mathbf{D}^{1 \times c} \end{array}$$

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→ "extract" an invertible codimension 1 matrix from Y

Lemma: if there exist $\mathbf{c} \in \mathbf{D}$, $\mathbf{v} \in \mathbf{D}^{1 \times (p+q'-1)}$, and $\mathbf{u} \in \mathbf{D}^{p+q'-1}$ s.t.

$$\mathbf{c} \sum_{j=1}^{q+p'} Y_{(q+p')j}^- Y_{j(q+p')} + \sum_{j=q+p'+1}^{r-1} \mathbf{v}_j \left(Y_{j(q+p')} + \mathbf{u}_j Y_{r(q+p')} \right) = 1 \quad (\bar{j} := j - (q+p'))$$

then, there exists a commutative diagram s.t. the top row is invertible

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Moreover, \mathbf{c} , \mathbf{v} , and \mathbf{u} exist as soon as

$$\begin{cases} 1 \leq \min(p + q', q + p') \\ \text{sr}(\mathbf{D}) \leq \max(p + q' - 1, q + p' - 1) \end{cases}$$

Ideas of the proof

Notations: consider $\ell \in \mathbf{D}^{1 \times (r-1)}$ and $F \in \mathbf{D}^{(r-1) \times r}$

$$\ell := (\mathbf{c}(Y'_1)_{(q+p')}, \mathbf{v}) \quad F := \begin{pmatrix} Y_1 \\ Y_2 + \mathbf{u}Y_3 \end{pmatrix}$$

where Y_1, Y_2, Y_3 (resp., Y'_1, Y'_2, Y'_3) have $q+p'$, $p+q'-1$, and 1 lines (resp., columns) and are s.t.

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \quad Y^- = \begin{pmatrix} Y'_1 & Y'_2 & Y'_3 \end{pmatrix}$$

Construction of the diagram: we define the matrices

$$G_1 := \begin{pmatrix} \text{id}_{r-1} - F_{\cdot, (q+p')\ell} \\ \ell \end{pmatrix} \quad H_1 := (\text{id}_{r-1} - F_{\cdot, (q+p')\ell} \quad F_{\cdot, (q+p')}) \quad W_1 := \begin{pmatrix} \text{id}_{q+p'} & 0 & 0 \\ 0 & \text{id}_{p+q'-1} & \mathbf{u} \\ 0 & 0 & 1 \end{pmatrix}$$

Commutativity/invertibility: matrix computations using the relation satisfied by \mathbf{c} , \mathbf{v} , and \mathbf{u}

Stable rank condition: lemma on quotients of free modules (NOT ALGORITHMIC!!!)

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Stable rank condition: lemma on quotients of free modules (**NOT ALGORITHMIC!!!**)

Removing many 0 lines

Notation: for $k \leq \min(q + p', p + q')$, let L_k and L'_k be defined by

$$L_k := \begin{pmatrix} R & 0 \\ 0 & \text{id}_{p'} \\ 0 & 0 \end{pmatrix} \leftarrow (p + q' - k) \qquad L'_k := \begin{pmatrix} 0 & 0 \\ \text{id}_p & 0 \\ 0 & R' \end{pmatrix} \leftarrow (q + p' - k)$$

Induction: assume that an invertible matrix $Y^{(k-1)}$ has been constructed s.t. $L_{k-1}X = Y^{(k-1)}L'_{k-1}$

Lemma: if $\exists c \in \mathbf{D}$, $\mathbf{v} \in \mathbf{D}^{1 \times (p+q'-k+1)}$, and $\mathbf{u} \in \mathbf{D}^{p+q'-k+1}$ s.t.

$$c \sum_{j=1}^{q+p'} \left(Y^{(k-1)} \right)_{(q+p'-k+1)j}^{-1} Y_{j(q+p'-k+1)}^{(k-1)} + \sum_{j=q+p'+1}^{r-k} \mathbf{v}_j \left(Y_{j(q+p'-k+1)}^{(k-1)} + \mathbf{u}_j Y_{(r-k+1)(q+p'-k+1)}^{(k-1)} \right) = 1$$

then, we can construct $Y^{(k)} := H_k W_k Y^{(k-1)} G_k$ invertible s.t. $L_k X = Y^{(k)} L'_k$. Moreover,

c , \mathbf{v} , and \mathbf{u} exist as soon as

$$\begin{cases} k \leq \min(p + q', q + p') \\ \text{sr}(\mathbf{D}) \leq \max(p + q' - k, q + p' - k) \end{cases}$$

Theorem [C., Cluzeau, Quadrat].

Let $\varphi : \mathbf{M} := \mathbf{D}^{1 \times p} / \mathbf{D}^{1 \times q} R \rightarrow \mathbf{M}' := \mathbf{D}^{1 \times p'} / \mathbf{D}^{1 \times q'} R$ be an iso. and let $z \in \mathbb{N}$ s.t.

$$z \leq \min(p + q', q + p') \quad \text{and} \quad \text{sr}(\mathbf{D}) \leq \max(p + q' - z, q + p' - z)$$

Then, the following matrices are equivalent

$$L_z := \begin{pmatrix} R & 0 \\ 0 & \text{id}_{p'} \\ 0 & 0 \end{pmatrix} \leftarrow (p + q' - z) \qquad L'_z := \begin{pmatrix} 0 & 0 \\ \text{id}_p & 0 \\ 0 & R' \end{pmatrix} \leftarrow (q + p' - z)$$

and there is an inductive procedure that constructs an invertible matrix $Y^{(z)}$ s.t. $L_z X = Y^{(z)} L'_z$.

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and there is an inductive procedure that constructs an invertible matrix $Y^{(z)}$ s.t. $L_z X = Y^{(z)} L'_z$.

Similarly if $i \in \mathbb{N}$ is s.t.

$$i \leq \min(p, p') \quad \text{and} \quad \text{sr}(\mathbf{D}) \leq \max(p - i, p' - i)$$

then, the following matrices are equivalent

$$\tilde{L} := \begin{pmatrix} R & 0 \\ 0 & \text{id}_{p'-i} \\ 0 & 0 \end{pmatrix} \leftarrow (p + q' - z) \qquad \tilde{L}' := \begin{pmatrix} 0 & 0 \\ \text{id}_{p-i} & 0 \\ 0 & R' \end{pmatrix} \leftarrow (q + p' - z)$$

and there is an inductive procedure that constructs invertible matrices \tilde{X}, \tilde{Y} s.t. $\tilde{L}\tilde{X} = \tilde{Y}\tilde{L}'$.

Example

Continue the previous example

$$R := \begin{pmatrix} \partial_1 & 0 \\ \frac{1}{2}\partial_2 & \frac{1}{2}\partial_1 \\ 0 & \partial_2 \end{pmatrix} \quad R' := \begin{pmatrix} \partial_1 & 0 & 0 \\ \partial_2 & -1 & 0 \\ 0 & \partial_1 & 0 \\ 0 & 1 & \partial_1 \\ 0 & 0 & \partial_2 \\ 0 & \partial_2 & 0 \end{pmatrix} \quad L := \begin{pmatrix} R & 0 \\ 0 & \text{id}_3 \\ 0 & 0 \end{pmatrix} \leftarrow 8$$

$$L' := \begin{pmatrix} 0 & 0 \\ \text{id}_2 & 0 \\ 0 & R' \end{pmatrix} \leftarrow 6$$

Theoretically: we can remove 5 lines of 0 and 0 identity block since

$$\rightarrow \text{sr}(\mathbf{D}) = \text{sr}(\mathbb{Q}[\partial_1, \partial_2]) = 3$$

$$\rightarrow z \leq \min(p + q', q + p') = 6 \quad \text{and} \quad \text{sr}(\mathbf{D}) \leq \max(p + q' - z, p + q' - z)$$

$$\rightarrow i \leq \min(p, p') = 2 \quad \text{and} \quad \text{sr}(\mathbf{D}) \leq \max(p - i, p' - i)$$

Practically: using obvious solutions and heuristics to compute \mathbf{c} , \mathbf{v} , and \mathbf{u} 's

\rightarrow we remove 5 lines of 0 and 2 identity blocks

Example

Continue the previous example

$$R := \begin{pmatrix} \partial_1 & 0 \\ \frac{1}{2}\partial_2 & \frac{1}{2}\partial_1 \\ 0 & \partial_2 \end{pmatrix}$$

$$R' := \begin{pmatrix} \partial_1 & 0 & 0 \\ \partial_2 & -1 & 0 \\ 0 & \partial_1 & 0 \\ 0 & 1 & \partial_1 \\ 0 & 0 & \partial_2 \\ 0 & \partial_2 & 0 \end{pmatrix}$$

$$L := \begin{pmatrix} R & 0 \\ 0 & \text{id}_3 \\ 0 & 0 \end{pmatrix} \leftarrow 8$$

$$L' := \begin{pmatrix} 0 & 0 \\ \text{id}_2 & 0 \\ 0 & R' \end{pmatrix} \leftarrow 6$$

Theoretically: we can remove **5 lines of 0** and **0 identity block** since

$$\rightarrow \text{sr}(\mathbf{D}) = \text{sr}(\mathbb{Q}[\partial_1, \partial_2]) = 3$$

$$\rightarrow z \leq \min(p + q', q + p') = 6 \quad \text{and} \quad \text{sr}(\mathbf{D}) \leq \max(p + q' - z, p + q' - z)$$

$$\rightarrow i \leq \min(p, p') = 2 \quad \text{and} \quad \text{sr}(\mathbf{D}) \leq \max(p - i, p' - i)$$

Practically: using obvious solutions and heuristics to compute **c**, **v**, and **u**'s

\rightarrow we remove 5 lines of 0 and 2 identity blocks

Example

Continue the previous example

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Example (end)

Summary: we obtained $\tilde{L}\tilde{X} = \tilde{Y}\tilde{L}'$ where

$$\tilde{L} := \begin{pmatrix} \partial_1 & \mathbf{0} & 0 \\ \frac{1}{2}\partial_2 & \frac{1}{2}\partial_1 & 0 \\ \mathbf{0} & \partial_2 & 0 \\ 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \tilde{L}' := \begin{pmatrix} 0 & 0 & 0 \\ \partial_1 & \mathbf{0} & \mathbf{0} \\ \partial_2 & -1 & \mathbf{0} \\ \mathbf{0} & \partial_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \partial_1 \\ \mathbf{0} & \mathbf{0} & \partial_2 \\ \mathbf{0} & \partial_2 & \mathbf{0} \end{pmatrix}$$

and the invertible matrices

$$\tilde{X} = \begin{pmatrix} 1 - \partial_2 & 1 & 0 \\ 0 & 0 & 1 \\ \partial_2 & -1 & 0 \end{pmatrix} \quad \tilde{X}^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ \partial_2 & 0 & -1 + \partial_2 \\ 0 & 1 & 0 \end{pmatrix} \quad \tilde{Y}, \tilde{Y}^{-1}$$

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Open question: is it possible to remove the last 0 line?

IV. CONCLUSION AND PERSPECTIVES

Conclusion and perspectives

We presented: a constructive approach to module theory

- ▷ matrix descriptions of module (iso)morphisms
- ▷ a constructive proof of Fitting's theorem
- ▷ a procedure that make the Warfield theorem almost constructive

Further work: make the proof of Warfield's theorem completely constructive

- ▷ at least for sufficiently nice rings/systems (ODE, (q) -differences, PDE)
- ▷ remove the last 0 line in the running example

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THANK YOU FOR LISTENING!