A Formal Description of Coherence in Abstract Rewriting

Algebraic rewriting seminar

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LIX

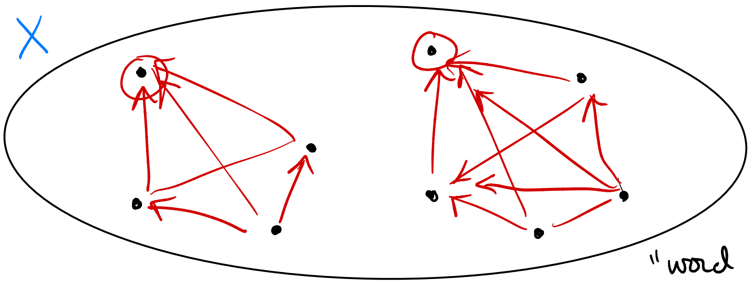
INTRODUCTION

↳ ARS, Relation algebras and coherence

Abstract rewriting theory studies directed transformations between abstract objects.

(x, y) $\qquad x \longrightarrow y$

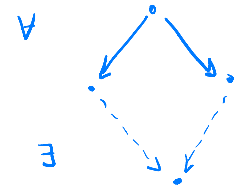
A first application of rewriting was in comparing elements of a set modulo some equivalence relation on a set X :



"word problem"

"Good" properties of an ARS, such as ...

CONFLUENCE $\qquad$ TERMINATION



$+ \quad \nexists\,(x_0 \to x_1 \to \cdots x_n \to \cdots)$

= CONVERGENCE

... express that in each equivalence class, there exists a distinguished element which represents the entire class. These are called normal forms.

In this way, comparison problems are solved by choosing a "base-point" in each connected component and describing a contraction of each to that point via directed transformations

"0-dimensional choice + direction"

RELATIONAL DESCRIPTION

An ARS may be thought of as a relation $R \subseteq X \times X$:

$$(x,y) \in R \quad \equiv \quad x \text{ is rewritten to } y$$

The set of relations on X, equipped with union and composition, form the relation algebra on X:

$$A \subseteq B \iff A \cup B = B.$$

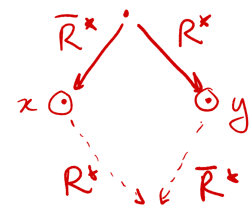
$$(\underline{P(X \times X)}, \cup, \emptyset, \circ, \mathbb{1}, \subseteq)$$

$\mathbb{1} \to X \times X$

- The converse of R is $\overline{R} = \{(y,x) \mid (x,y) \in R\}$
- The reflexive, transitive closure of a relation S is denoted S^*.

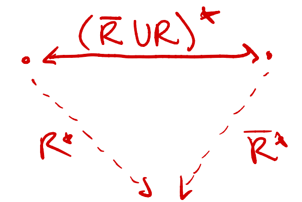
Properties of the ARS become inequalities in this algebra:

CONFLUENCE



$$\overline{R}^* \circ R^* \subseteq R^* \circ \overline{R}^*$$

CHURCH-ROSSER



$$(R \cup \overline{R})^* \subseteq R^* \circ \overline{R}^*.$$

Classic results may be expressed and proved in this setting, using formal methods:

Thm (Church-Rosser)
Let $R \subseteq P(X \times X)$. Then

$$\overline{R}^* \circ R^* \subseteq R^* \circ \overline{R}^* \iff (R \cup \overline{R})^* \subseteq R^* \circ \overline{R}^*$$

⚠ In the relational description, there is no clean notion of "parallel" transformation.

Indeed, for $R, S \subseteq P(X \times X)$ we may have

$(x,y) \in R$ and $(x,y) \in S$, but $R \neq S$.

We have limited 1-dimensional information...

POLYGRAPHIC DESCRIPTION

We can also describe an ARS via the notion of 1-polygraph $\Sigma = (\Sigma_0, \Sigma_1)$:

$$
\begin{tikzcd}
\Sigma_0 & & \Sigma_1^* \arrow[ll, shift left] \arrow[ll, shift right] \\
\Sigma_1^T \arrow[u, shift left] \arrow[u, shift right] & & \Sigma_1 \arrow[ll] \arrow[u] \arrow[ull, shift left, "s_0"'] \arrow[ull, shift right, "t_0"]
\end{tikzcd}
$$

free groupoid

free category

$$x \underset{g}{\overset{f}{\rightrightarrows}} y$$

$\equiv$

x is rewritten to y via f.

The same properties and results may be expressed here, but now we can compare the paths

String rewriting

or

Coherence for ARS

Coherence properties are tracked via higher-dimensional cells; we consider a cellular extension A of Σ^T:

$$
\begin{tikzcd}
\Sigma_0 & \Sigma_1^T \arrow[l, shift right, "s_0"'] \arrow[l, shift left, "t_0"] & A^T \arrow[l, shift right, "s_1"'] \arrow[l, shift left, "t_1"] \\
& \Sigma_1 \arrow[ul, shift left, "s_0"'] \arrow[ul, shift right, "t_0"] \arrow[u] & A \arrow[ul, shift left, "s_1"'] \arrow[ul, shift right, "t_1"] \arrow[u]
\end{tikzcd}
$$

free 2-groupoid.

The source and target maps satisfy globularity:

$s_0 s_1 = s_0 t_1$ and $t_0 s_1 = t_0 t_0$.

$$
\begin{tikzcd}
\cdot \arrow[r, bend left=50, "s_1(\alpha)"{name=U}] \arrow[r, bend right=50, "t_1(\alpha)"'{name=D}] \arrow[from=U, to=D, phantom, "\Downarrow \alpha \in A"] & \cdot
\end{tikzcd}
$$

While convergence deals with 0-dimensional contractibility along directed paths, coherence expresses 1-dimensional contractibility along undirected 2-cells:

DEF: an ARS Σ is coherent wrt A if:

$$\forall f, g \in \Sigma_1^+ ,\ \substack{s_0(f) = s_0(g) \\ t_0(f) = t_0(g)} ,\quad \exists \alpha \in A^\top ,\ \alpha : f \Rightarrow g .$$

f and g are parallel

$$\begin{tikzcd} \bullet \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow \alpha"] & \bullet \end{tikzcd}$$

In order to prove coherence of a convergent ARS, we proceed in a similar way to before; convergence is used to show that we may choose a (1-dimensional) "base-point" in each set of parallel arrows.

This is the notion of normalisation strategy.

DEF: Recall that an ARS generates an equivalence relation on 0-cells (connected components). Given a section of that relation

$$\widehat{(-)} : \Sigma_0 \longrightarrow \Sigma_0 ,$$

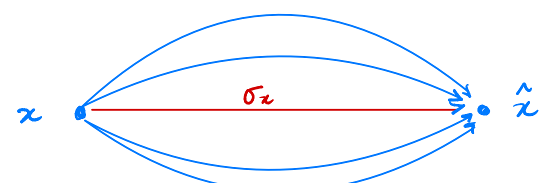
a normalisation strategy for Σ wrt $\widehat{(-)}$ is a map

$$\sigma : \Sigma_0 \longrightarrow \Sigma_1^*$$

such that for all $x \in \Sigma_0$

$$\sigma_x : x \longrightarrow \hat{x} \qquad \text{and} \qquad \sigma_{\hat{x}} = 1_{\hat{x}}$$

Remark: In the case of a convergent ARS, we may consider the section given by normal forms.

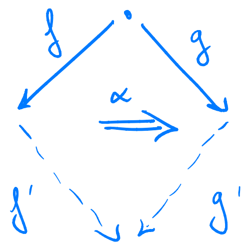


For a convergent ARS Σ, coherence wrt the cellular extension A given by local branchings can be proved in three steps:

For every local branching (f,g), choose a confluence (f',g').

$f,g \in \Sigma_1$, $s_0(f)=s_0(g)$

Define A by taking a 2-cell $\alpha : f\star f' \Rightarrow g \star g'$ for every local branching.



$f',g' \in \Sigma_1^*$

$t_0(f')=t_0(g')$

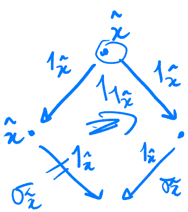
Fix a normalisation strategy σ wrt normal forms.

$\sigma : \Sigma_0 \to \Sigma_1^*$

(1) Normalising Newman

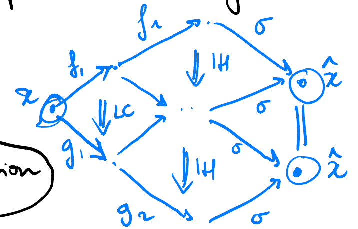
"Every branching can be paved to a confluence in σ"

base case



$1_{\hat{x}} = \sigma_{\hat{x}}$

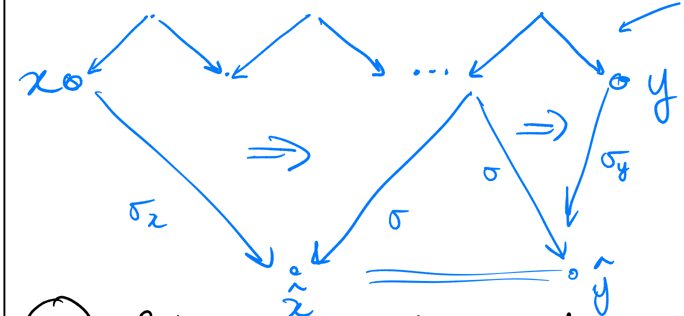
induction step



(2) Normalising Church-Rosser:

"Every zig-zag can be paved to a confluence in σ"

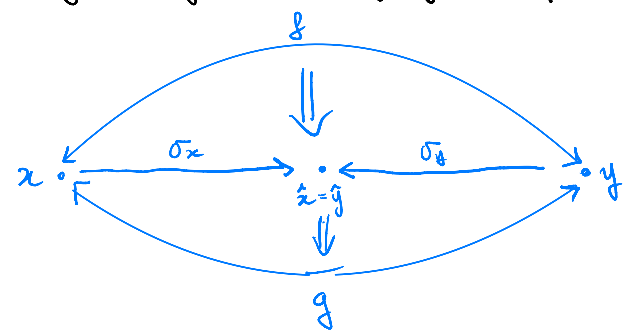
$f \in \Sigma_1^\top$



$\sigma_x \star_0 \sigma_y^-$

(3) Coherence via base-points:

"Every pair of parallel zig-zags are paved"



GOAL Formulate and prove the coherence thm for convergent ARS in Kleene algebras.

Outline:

1. modal Kleene algebras and ARS
2. 2-Kleene algebras and pairing
3. Strategies and coherence.

1. MKA & ARS.

A Kleene algebra is a structure

$$(K, +, 0, \cdot, 1, (-)^*)$$

such that...

→ $(K, +, 0)$ is a commutative idempotent monoid

$\varphi + \varphi = \varphi \quad \forall \varphi \in K$ "union"

⚠ This endows K with an ordering

$\varphi \leq \psi \iff \varphi + \psi = \psi$

→ $(K, \cdot, 1)$ is a monoid "composition/concatenation"

→ Multiplication distributes over addition

→ $(-)^* : K \longrightarrow K$ is a map satisfying:

$1 + \varphi \cdot \varphi^* \leq \varphi^*$, $\quad 1 + \varphi^* \cdot \varphi \leq \varphi^*$ (unfold)

- $\xi + \varphi\psi \leq \psi \implies \varphi^* \cdot \xi \leq \psi$
- $\xi + \psi\varphi \leq \psi \implies \xi \cdot \varphi^* \leq \psi$

(induction)

This is called the Kleene star.

Example : Path algebras

Let Σ be a 1-polygraph. Then

$$(\mathcal{P}(\Sigma_1^\top), \cup, \emptyset, \odot_0, \mathbb{1}_0, (-)^{*_0})$$

is a Kleene algebra, denoted by $K(\Sigma)$.

$$\varphi \odot_0 \psi := \{ f *_0 g \mid f \in \varphi, g \in \psi \text{ and } t_0(f) = s_0(g) \},$$

$$\varphi^{*_0} := \{ f_1 *_0 \cdots *_0 f_n \mid n \in \mathbb{N}, f_i \in \varphi^*, t_0(f_i) = s_0(f_{i+1}) \} \cup \{ 1_x \mid x \in \Sigma_0 \}$$

$$\mathbb{1}_0 = \{ 1_x \mid x \in \Sigma_0 \}.$$

Modal Kleene algebras

Consider a Kleene algebra equipped with a map $d : K \longrightarrow K$ satisfying

$$d(0) = 0, \qquad \varphi = d(\varphi) \cdot \varphi, \qquad d(\varphi\psi) \leq d(\varphi\, d(\psi)),$$

$$d(\varphi) \leq 1, \qquad d(\varphi + \psi) = d(\varphi) + d(\psi).$$

This is called a domain operation. The set

$$K_0 = \{ \varphi \mid d(\varphi) = \varphi \} = d(K)$$

is called the domain algebra; restricting the operations to K_0 gives a distributive lattice bounded by 0 and 1.

We may also consider a range operation $r : K \longrightarrow K$ satisfying domain axioms in the opposite Kleene algebra (in which multiplication is reversed).

G. Struth et al

A Kleene algebra with a domain and a range is called a modal Kleene algebra if

$$d \circ r = r \quad \text{and} \quad r \circ d = d.$$

This assumes that the domain and range algebras coincide.

N.B. Elements of K_0 will be denoted $p, q, r, \dots$

$ad\, ad = d$

⚠ Axiomatising notions of antidomain and antirange, we may equip K_0 with a Boolean complementation, i.e. $\boxed{\neg : K_0 \to K_0}$

$$p \cdot \neg p = 0, \qquad p + \neg p = 1.$$

These are called Boolean MKAs. Henceforth we will consider such structures.

Example: For $\varphi \in K(\Sigma)$, we have:

$$\longrightarrow \; d(\varphi) = \{ 1_x \mid \exists f \in \varphi,\ s_0(f) = x \},$$

$$r(\varphi) = \{ 1_y \mid \exists f \in \varphi,\ t_0(f) = y \},$$

$$p = \{ 1_x \mid x \in A \subseteq \Sigma_0 \}. \qquad \neg p = \{ 1_y \mid y \in \Sigma_0 \setminus A \}.$$

$$K(\Sigma)_0 \cong \mathcal{P}(\Sigma_0).$$

Modalities

As the name indicates, in a modal Kleene algebra, each element gives rise to modal diamond operators on K_0:

For every $\varphi \in K$, $|\varphi\rangle : K_0 \to K_0$

$$|\varphi\rangle(p) := d(\varphi \cdot p) \qquad \text{"forward diamond"}$$

"The set of points from which a φ-step goes into p"

Similarly, we define a backward diamond

$$\langle \varphi |(p) = r(p \cdot \varphi),$$

and forward/backward box operators:

$$|\varphi](p) = \neg |\varphi\rangle(\neg p) \quad \text{and} \quad [\varphi|(p) = \neg \langle \varphi |(\neg p).$$

Converses in MKA.

Let K be an MKA equipped with an involution

$$\overline{(-)} : K \longrightarrow K \qquad \text{"converse"}$$

satisfying

$$\overline{(\varphi + \psi)} = \overline{\varphi} + \overline{\psi}, \qquad \overline{\varphi \cdot \psi} = \overline{\psi} \cdot \overline{\varphi},$$

$$\overline{(\varphi^*)} = (\overline{\varphi})^* \quad \text{and} \quad d(\varphi) \leq \varphi \cdot \overline{\varphi}.$$

Conversion exchanges domain/range as well as forward/backward modalities:

$$d(\varphi) = r(\overline{\varphi}) \qquad |\varphi\rangle = \langle \overline{\varphi}|$$

$$r(\varphi) = d(\overline{\varphi}) \qquad \langle \varphi | = |\overline{\varphi}\rangle.$$

Finally, it is idempotent on K_0:

$$\overline{p} = p \quad \text{for all } p \in K_0.$$

Example: For $\varphi \in K(\Sigma)$, $p \in K(\Sigma)_0$,

$$|\varphi\rangle(p) = \{ 1_x \mid \exists (f : x \to y) \in \varphi \text{ with } 1_y \in p \}.$$

$$\overline{\varphi} = \{ f^{-1} \mid f \in \varphi \}.$$

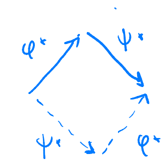
Abstract rewriting in MKAs.

- Confluence properties are given by semi-commutation. Let $\varphi, \psi \in K$:

$\longrightarrow$ (φ, ψ) semi-commutes

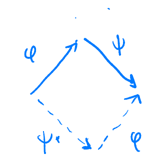
- (globally) if

$$\varphi^* \cdot \psi^* \leq \psi^* \cdot \varphi^*$$



- locally if

$$\varphi \cdot \psi \leq \psi^* \varphi^*$$



We say that φ is (locally) confluent if $(\overline{\varphi}, \varphi)$ semi-commutes (locally).

→ (φ, ψ) has the Church-Rosser property if

$$(\varphi + \psi)^* \leq \psi^* \varphi^*$$

$$\begin{tikzcd} \cdot \arrow[rr, "(\varphi+\psi)^*"] \arrow[dr, dashed, "\psi^*"'] & & \cdot \\ & \cdot \arrow[ur, dashed, "\varphi^*"'] & \end{tikzcd}$$

φ has the CR property if $(\bar{\varphi}, \varphi)$ does.

- Termination properties are captured via modal operators. Let $\varphi \in K$:

φ terminates if for all $p \in K_0$,

$$\underbrace{p \leq |\varphi\rangle(p)}_{\text{"p contains a } \varphi\text{-loop" or a } \infty\text{-}\varphi\text{-path}} \implies \underbrace{p = 0}_{\text{"p is empty"}}.$$

We have an equivalent formulation with boxes:

$$\forall p \in K_0, \quad |\varphi](p) \leq p \implies p = 1.$$

When $\varphi \in K$ is confluent and terminates, we say that φ is convergent.

Classic results from abstract rewriting may now be expressed and proved in MKA:

> **Theorem** (Formal Newman) G. Struth et al.
> Let $\varphi \in K$ be locally confluent and terminating. Then φ is (globally) confluent.

> **Theorem** (Formal Church-Rosser) G. Struth
> Let $\varphi, \psi \in K$. Then
> (φ, ψ) semi-commutes $\iff$ (φ, ψ) has the CR prop.

I can sketch a proof if there is interest.
Otherwise on to higher dimensions!

2. 2-MKAs & pairing.

In order to capture coherence properties in Kleene algebra, we need an extra dimension.

A modal 2-Kleene algebra (2-MKA) is a structure

$$(K, +, 0, \overline{(-)}, \odot_i, 1_i, (-)^{*_i}, d_i, r_i)_{i=0,1}$$

such that

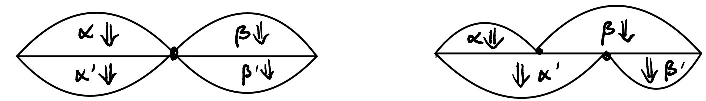
- for $i=0$, we obtain a Boolean MKA with converse.
- for $i=1$ we obtain a MKA.

The domain algebra associated with d_i will be denoted by K_i.

Now we need some axioms describing the interaction of these structures:

i) Weak exchange law: for $A, B, A', B' \in K$,

$$(A \odot_1 A') \odot_0 (B \odot_1 B') \leq (A \odot_0 B) \odot_1 (A' \odot_0 B')$$



ii) Completeness of 1-unit wrt 0-mult:

$$1_1 \odot_0 1_1 = 1_1$$

iii) Domain/range absorption: $K_0 \subseteq K_1$

$$d_1 \circ d_0 = d_0 \quad \text{and} \quad r_1 \circ r_0 = r_0$$

iv) Kleene star axioms: for $A \in K$, $\varphi \in K_1$,

$$\varphi \odot_0 A^{*_1} \leq (\varphi \odot_0 A)^{*_1} \quad \text{and} \quad A^{*_1} \odot_0 \varphi \leq (A \odot_0 \varphi)^{*_1}$$

⚠ To avoid confusion, elements of …

… K_0 will be denoted $p, q, r, \dots$

… K_1 will be denoted $\varphi, \psi, \xi, \dots$

Other elements of K will be denoted $A, B, C, \dots$

Finally, we want domain and range to satisfy globularity conditions :

$$d_0 = \underbrace{d_0 \circ d_1 = d_0 \circ r_1} \qquad \underbrace{r_0 \circ d_1 = r_0 \circ r_1} = r_0$$

$$d_1(A \odot_0 B) = d_1(A) \odot_0 d_1(B)$$

$$r_1(A \odot_0 B) = r_1(A) \odot_0 r_1(B)$$

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\alpha"] & \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\beta"] & \bullet \end{tikzcd}$$

This means that we have homomorphisms

$$\begin{tikzcd} K_0 \arrow[r, shift left=2] & K_1 \arrow[l, shift left=4, "d_0"'] \arrow[l, shift right=2, "r_0"] \arrow[r, shift left=2] & K \arrow[l, shift left=4, "d_1"'] \arrow[l, shift right=2, "r_1"] \end{tikzcd}$$

MKA   MKA

Example : 2-path algebra.

Let Σ be a 1-polygraph with cellular extension A. Then the structure

$$\left(\mathcal{P}(A_2^\top), \cup, \emptyset, \odot_i, \mathbb{1}_i, (-)^{*_i}, d_i, r_i\right)_{i=0,1}$$

is a globular 2-MKA, where the operations are defined as in the 1-polygraph case.

This structure is denoted by $K(\Sigma, A)$.

Modalities

Just as in the case of MKA, we obtain modal operators in each dimension:

$$\left[\begin{array}{ll} |A\rangle_1(\varphi) = d_1(A \odot_1 \varphi) & , \ \langle A|_1(\varphi) = r_1(\varphi \odot_1 A) \\ |A\rangle_0(p) = d_0(A \odot_0 p) & , \ \langle A|_0(p) = r_0(A \odot_0 p). \end{array}\right.$$

Pairing in 2-MKA.

Let $A \in K$ and $\varphi, \psi \in K_1$.

- A is a local confluence filler for (φ,ψ) if

$$\varphi \odot_0 \psi \leq |A\rangle_1(\psi^{*_0} \odot_0 \varphi^{*_0}) = d_1(A \odot_1 (\psi^{*_0} \odot_0 \varphi^{*_0}))$$

$$\begin{tikzcd} & \cdot \arrow[dr, "\psi"] & \\ \cdot \arrow[ur, "\varphi"] \arrow[dr, dashed, "\psi^*"'] & \Downarrow A & \cdot \arrow[dl, dashed, "\varphi^*"] \\ & \cdot & \end{tikzcd} \quad \begin{matrix} \forall \\ \\ \exists \end{matrix}$$

- A is a confluence filler for (φ,ψ) if

$$\varphi^{*_0} \odot_0 \psi^{*_0} \leq |A\rangle_1(\psi^{*_0} \odot_0 \varphi^{*_0})$$

$$\begin{tikzcd} & \cdot \arrow[dr, "\psi^*"] & \\ \cdot \arrow[ur, "\varphi^*"] \arrow[dr, dashed, "\psi^*"'] & \Downarrow A & \cdot \arrow[dl, dashed, "\varphi^*"] \\ & \cdot & \end{tikzcd}$$

- A is a Church-Rosser filler for (φ,ψ) if

$$(\varphi+\psi)^{*_0} \leq |A\rangle_1(\psi^{*_0} \odot_0 \varphi^{*_0})$$

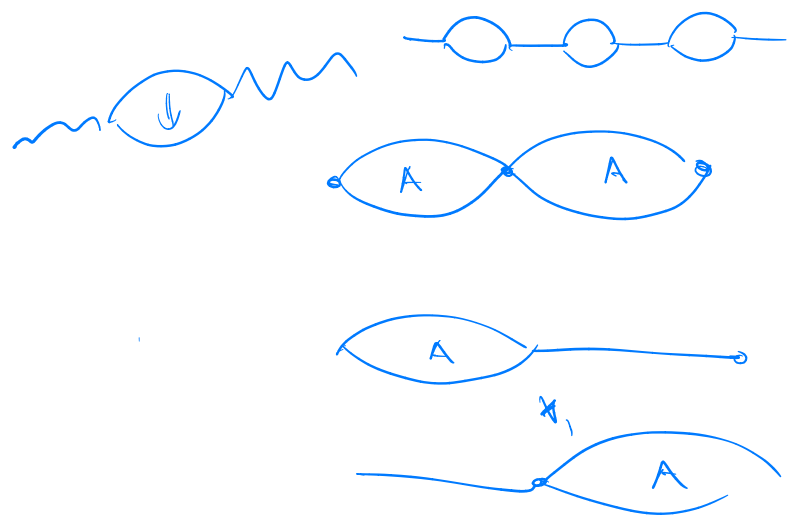
$$\begin{tikzcd} \cdot \arrow[rr, leftrightarrow, "(\varphi+\psi)^*"] \arrow[dr, dashed, "\psi^*"'] & \Downarrow A & \cdot \\ & \cdot \arrow[ur, dashed, "\varphi^*"'] & \end{tikzcd}$$

Completion

Let A be a filler for (φ,ψ). We define the total whiskering of A by

$$\hat{A} = (\varphi+\psi)^{*_0} \odot_0 A \odot_0 (\varphi+\psi)^{*_0}.$$

The completion of A is the element $\hat{A}^{*_1}$.



3. Strategies & coherence

Notice that in a 2-MKA, we can't describe coherence via pairing from zig-zags to zig-zags; indeed

$\forall f, g \in \Sigma_1^{\dagger}$ parallel $\exists \alpha : f \Rightarrow g$.

$$\underbrace{|\hat{A}^{*_1}>_1}((\varphi+\overline{\varphi})^{*_0}) \geqslant (\varphi+\overline{\varphi})^{*_0}$$

is satisfied by any element A since

$$\hat{A}^{*_1} \geqslant 1_1$$

The solution to this problem is to capture the notion of normalisation strategy in MKAs; coherence will be satisfied when

$$(\varphi+\overline{\varphi})^{*_0} \leq |\hat{A}^{*_1}>_1 (\sigma \odot_0 \overline{\sigma})$$

$$\begin{tikzcd} \cdot \arrow[rr, leftrightarrow, "(\varphi+\overline{\varphi})^{*_0}"] \arrow[dr, dashed, "\sigma"'] & \arrow[d, phantom, "\Downarrow A^{*_1}"] & \cdot \\ & \cdot \arrow[ur, dashed, "\overline{\sigma}"'] & \end{tikzcd}$$

$\sigma_x : x \to \tilde{x}$

Sections, skeleta & strategies

Let K_1 be a Boolean MKA and $\varphi \in K_1$, $p \in K_0$.

→ The **equivalence** generated by φ is the element $\varphi^{\top} := (\varphi+\overline{\varphi})^{*_0}$.

→ p is a **covering set** for φ if

$$|\varphi^{\top}>(p) = 1_0$$

A **section** of φ is a minimal covering set.

→ A **wide sub** of φ is an element $\psi \in K_1$ such that $\psi \leq \varphi$ and

$$|\varphi>_0 = |\psi>_0 \quad \text{and} \quad <\varphi|_0 = <\psi|_0 .$$

A **skeleton** of φ is a minimal wide sub.

→ Finally, given a section s of φ, a **strategy** for (φ, s) is a skeleton σ of $\varphi^{\top} \odot_0 s$ such that $s \odot_0 \sigma \leq s$.

$\sigma_{\hat{x}} = 1_{\hat{x}}$

Exhaustive iteration & normal forms.

Let $\varphi \in K_1$. The exhaustive iteration of φ is defined as

$$exh(\varphi) := \varphi^{*_0} \odot_0 (\neg d_0(\varphi)).$$

"do φ steps until no longer possible".

The normal forms element of φ is defined as

$$nf_\varphi := r_0(exh(\varphi))$$

We rediscover known properties:

→ If φ is confluent, normal forms are unique:

$$\langle exh(\varphi) |_0 | exh(\varphi) \rangle_0 (p) \leq p$$

→ If φ terminates, a normal form may be reached from any point: $d(exh(\varphi)) = 1_0$.

Even better, we have:

$\boxed{\begin{array}{l} \underline{\text{Lemma}}\text{: For } \varphi \in K_1 \text{ convergent,} \\ \quad \text{i) } nf_\varphi \text{ is a section of } \varphi. \\ \quad \text{ii) any skeleton of } exh(\varphi) \text{ is a} \\ \quad\quad \text{strategy for } (\varphi, nf_\varphi). \end{array}}$

This result, and other technical lemmas, suggest that this is a "good" definition of strategy in MKA.

In the polygraphic model $K(\Sigma)$, a strategy $\sigma : \Sigma_0 \longrightarrow \Sigma_1^T$ gives a strategy in the sense of MKA by simply considering its image:

$$\sigma(\Sigma_0) = \{\sigma_x \mid x \in \Sigma_0\}.$$

A formal coherence theorem.

Now we can state the main results of this work:

Theorem (Coherent normalising Newman).

Let K be a 2-MKA and $\varphi \in K_1$ convergent. If A is a local confluence filler for φ and σ is a strategy for $(\varphi, \mathrm{nf}_\varphi)$, then

$$\overline{\varphi}^{*_0} \odot_0 \varphi^{*_0} \leq |\hat{A}^{*_1}\rangle_1 (\sigma \odot_0 \bar{\sigma})$$

$$\begin{tikzcd} & \cdot \arrow[dr, "\varphi^{*_0}"] \arrow[dd, phantom, "\Downarrow \hat{A}^{*_1}"] & \\ \cdot \arrow[ur, "\overline{\varphi}^{*_0}"] \arrow[dr, "\sigma"'] & & \cdot \\ & \cdot \arrow[ur, "\bar{\sigma}"'] & \end{tikzcd}$$

This is ① from the intro.

Theorem (Abstract coherence).

Under the same hypotheses, we have

$$\varphi^T \leq |\hat{A}^{*_1}\rangle_1 (\sigma \odot_0 \bar{\sigma}).$$

$$\begin{tikzcd} \cdot \arrow[rr, leftrightarrow, "\varphi^T"] \arrow[dr, "\sigma"'] & \arrow[d, phantom, "\Downarrow \hat{A}^{*_1}"] & \cdot \\ & \cdot \arrow[ur, "\bar{\sigma}"'] & \end{tikzcd}$$

This is ② from intro.

THANK YOU!