String Rewriting Systems

Séminaire de réécriture, séance 1

14 Janvier 2021

Abstract rewriting in algebraic contexts

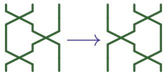
- Abstract rewriting appears in various algebraic contexts:
  - String rewriting systems,

$$abcde \longrightarrow cdeba$$

  - Term rewriting systems,

$$\begin{array}{lll} \rho_1 : & e \cdot x & \rightarrow & x \\ \rho_2 : & I(x) \cdot x & \rightarrow & e \\ \rho_3 : & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \end{array}$$

  - Diagrammatic rewriting systems,



  - In higher-dimensional categories,
  - In linear structures, such as associative algebras:

$$xyz \longrightarrow 2x^3 + 2y^3 + 2z^3$$

  - In operads.
- Algebraic tools provide ways to prove termination and confluence:
  - Monomial orders to prove termination.
  - Local confluence is examined by confluence of critical branchings.

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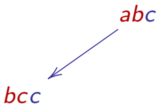

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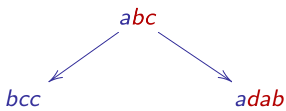

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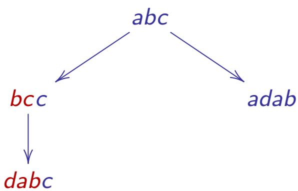

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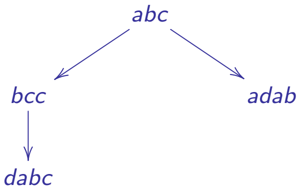

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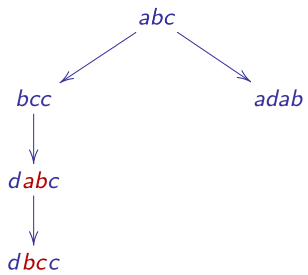

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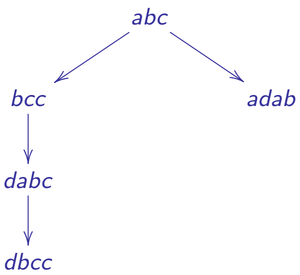

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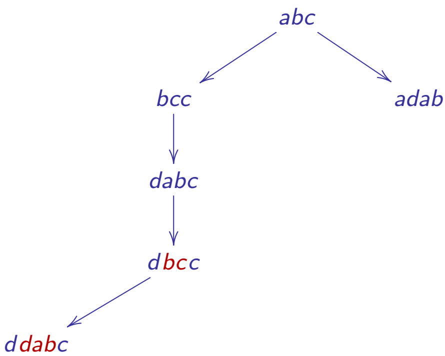

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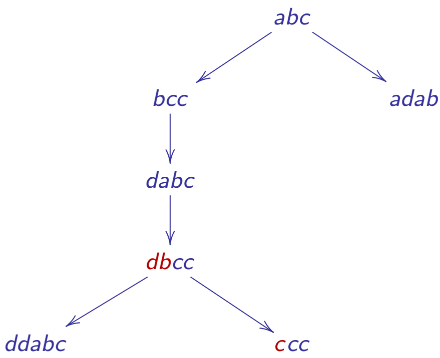

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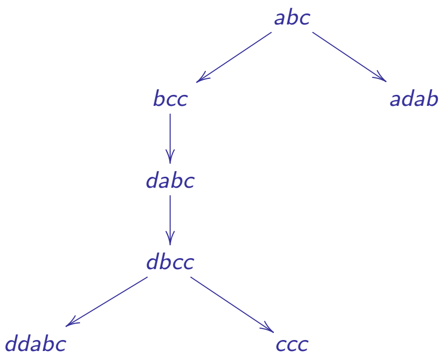

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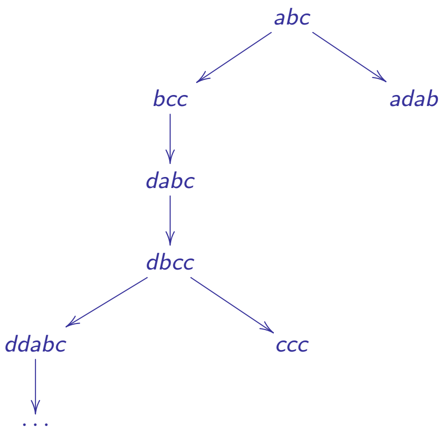

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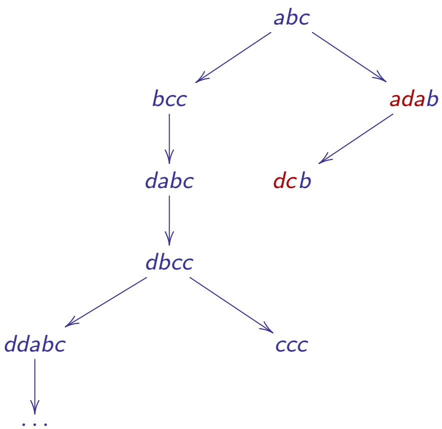

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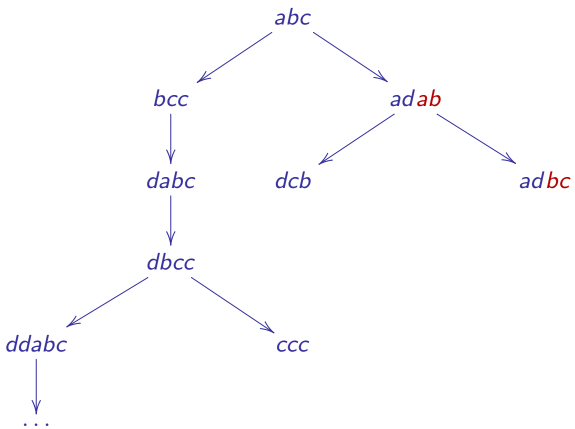

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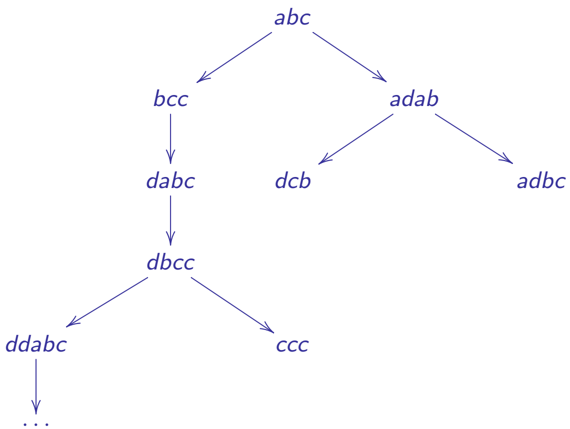

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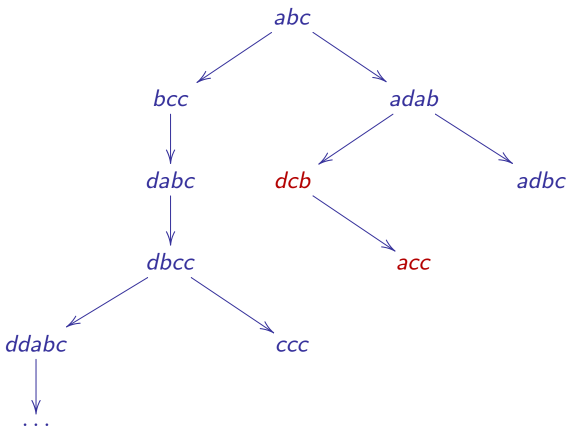

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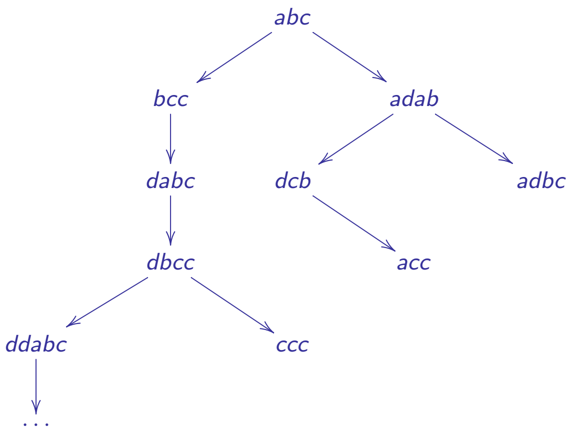

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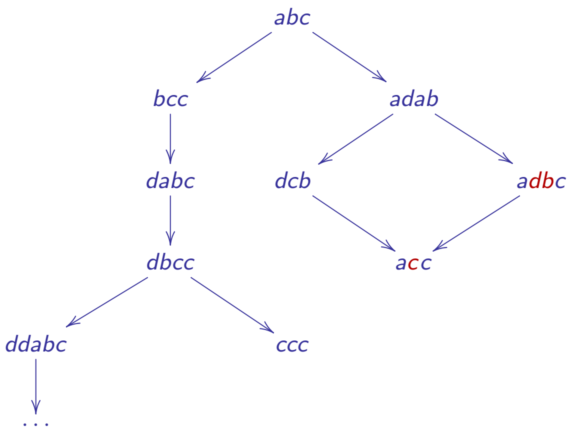

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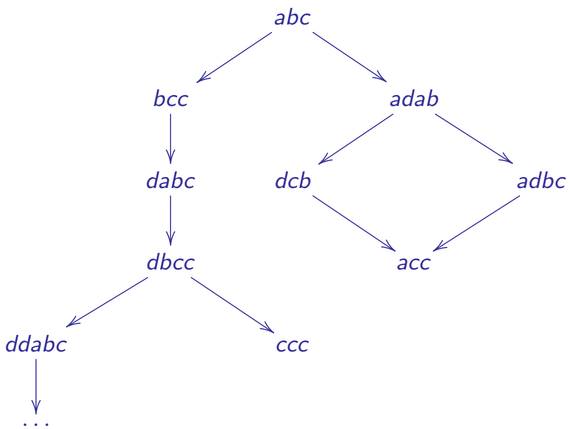

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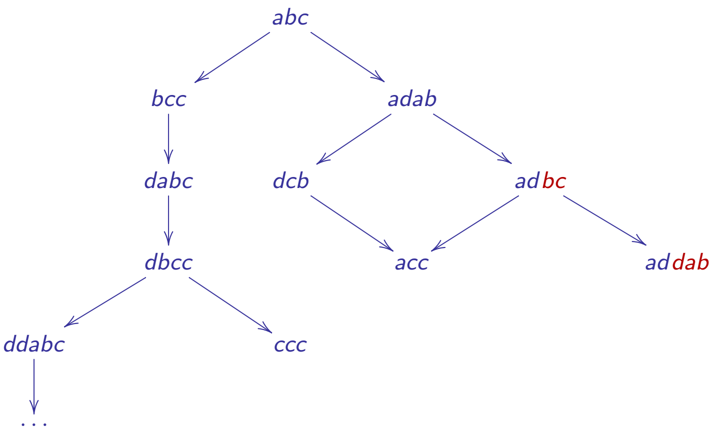

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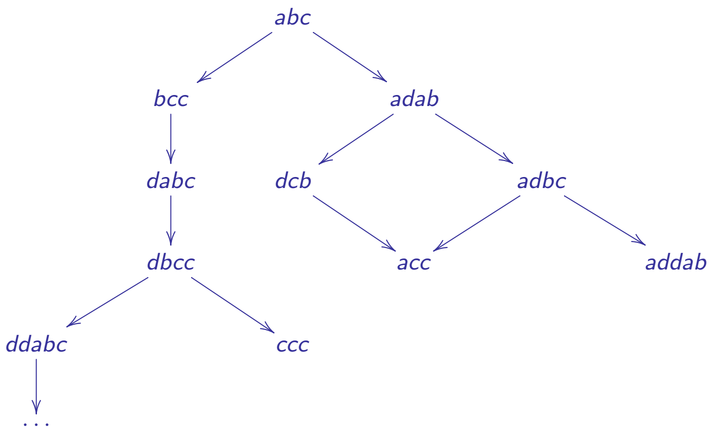

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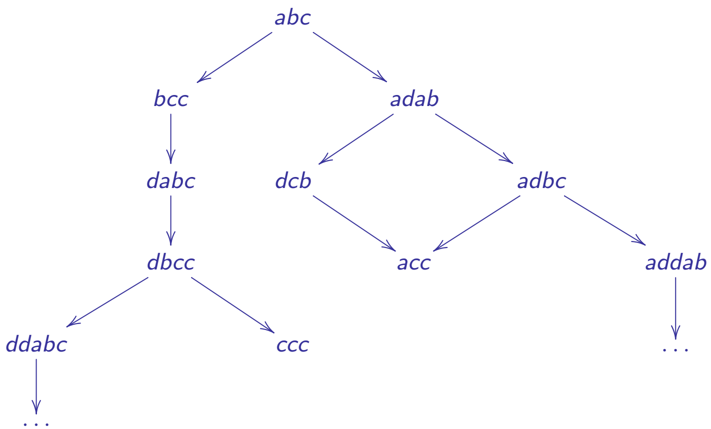
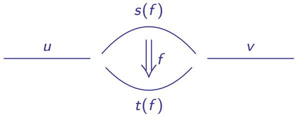


Termination, normal forms, branchings

- Let (X, R) be a SRS. A rewriting step of (X, R) is a reduction



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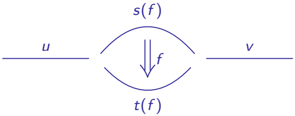

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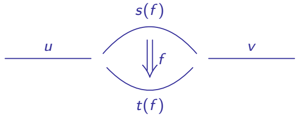

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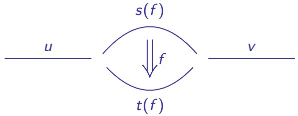

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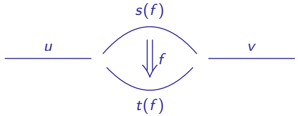

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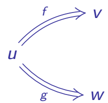
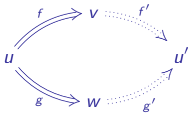
- A branching (resp. local branching) of (X, R) is:



where f, g are rewriting sequences/paths (resp. rewriting steps), and $u, v, w \in X^*$.

Confluence and branchings

- A (local) branching (f, g) is confluent is there exists rewriting paths f' and g' as follows:



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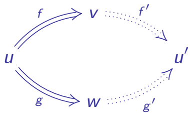

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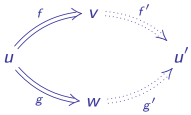

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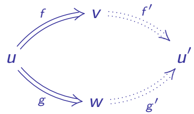

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- A SRS is convergent if it is both terminating and confluent: any element $x \in X^*$ admits exactly one normal form.

- **Newman's lemma:** If (X, R) terminates, confluence and local confluence are equivalent.

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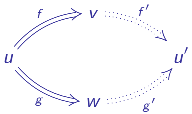
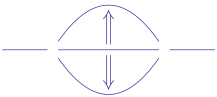
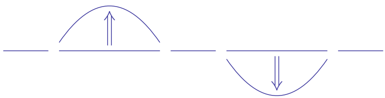
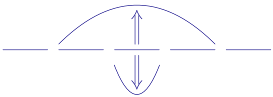
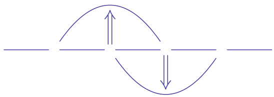


- If a SRS is confluent, any element $x \in X^*$ admits at most one normal form.
- A SRS is convergent if it is both terminating and confluent: any element $x \in X^*$ admits exactly one normal form.
- **Newman's lemma:** If (X, R) terminates, confluence and local confluence are equivalent.
- There are three forms of local branchings:
  - Aspherical: 
  - Peiffer/orthogonal: 
  - Overlappings:  and 

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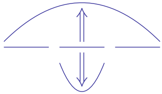
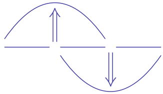
 and  .

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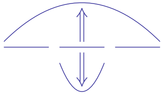

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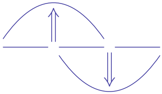

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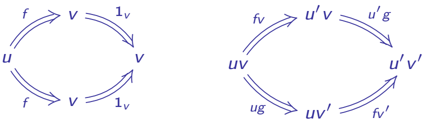

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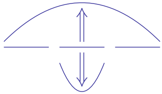

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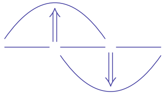

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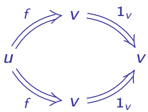

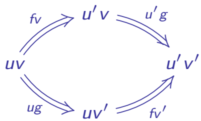

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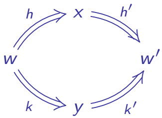

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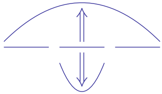

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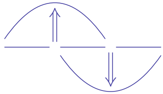

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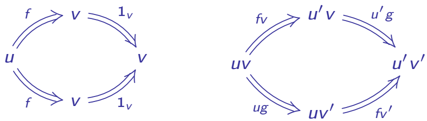

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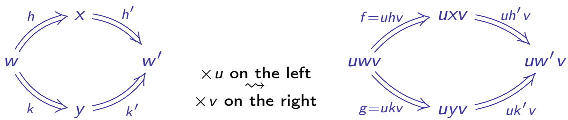

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- **How to prove confluence ?**
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  - Then, study the confluence of all critical branchings.

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Example. $X = \{a, b\}$ and $R = \{ab \overset{\alpha}{\Rightarrow} ba\}$.

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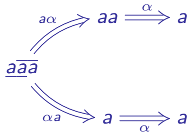

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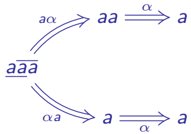

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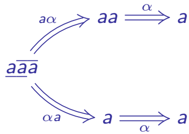

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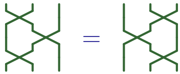
Exemple. $X = \{s, t\}$ et $R = \{sts \overset{\alpha}{\Rightarrow} tst\}$. $s =$  $t =$  

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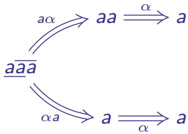

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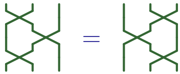
Exemple. $X = \{s, t\}$ et $R = \{sts \overset{\alpha}{\Rightarrow} tst\}$. $s =$  $t =$  

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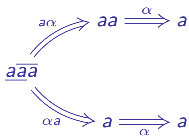

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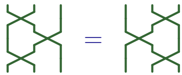
Exemple. $X = \{s, t\}$ et $R = \{sts \overset{\alpha}{\Rightarrow} tst\}$. $s =$  $t =$  

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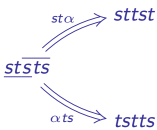

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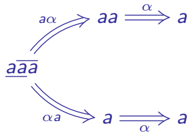

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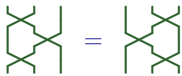
Exemple. $X = \{s, t\}$ et $R = \{sts \overset{\alpha}{\Rightarrow} tst\}$. $s =$  $t =$  

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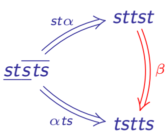

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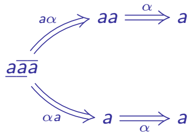

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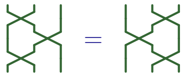
Exemple. $X = \{s, t\}$ et $R = \{sts \overset{\alpha}{\Rightarrow} tst\}$. $s =$  | $t =$  

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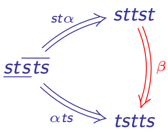

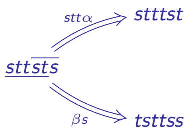

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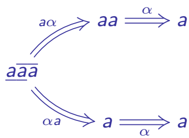

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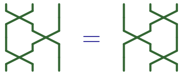
Exemple. $X = \{s, t\}$ et $R = \{sts \overset{\alpha}{\Rightarrow} tst\}$. $s =$  $t =$  

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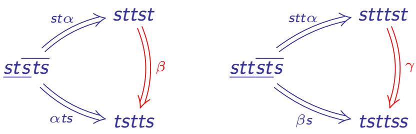

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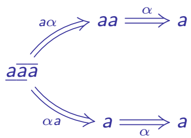

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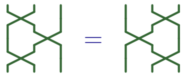

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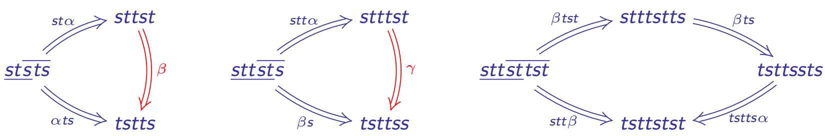

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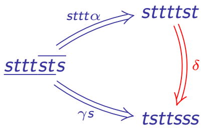

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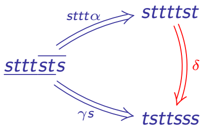

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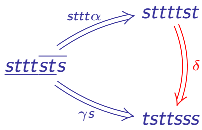

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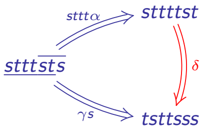

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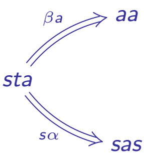

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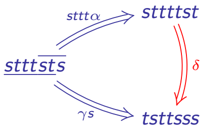

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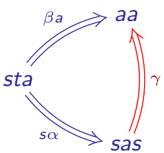

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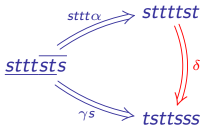

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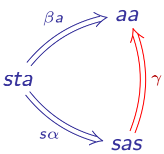

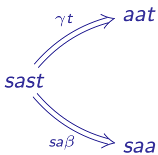

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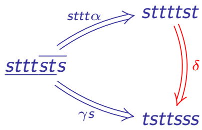

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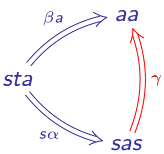

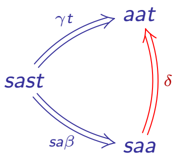

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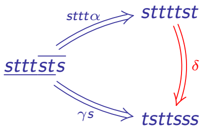

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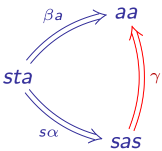

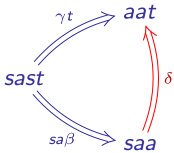

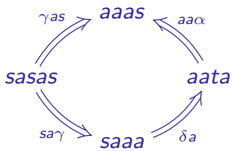

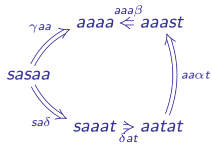

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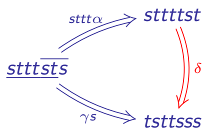

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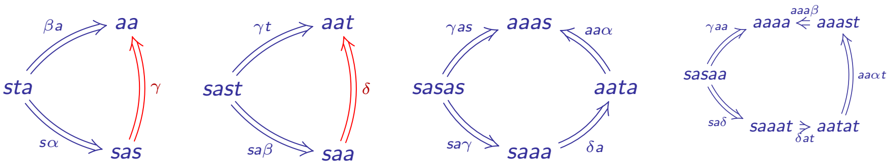

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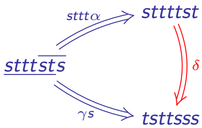

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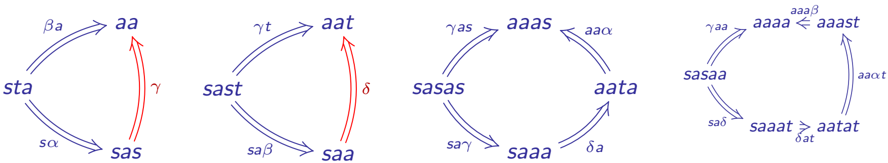

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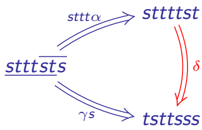

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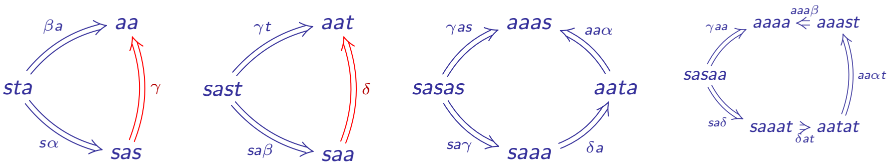

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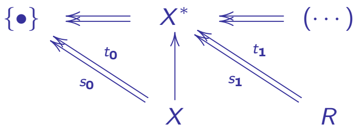
- **Example:** $stasas \overset{?}{=} sasssts$

$$\underline{st}a\underline{sas} \Rightarrow aaaa \Leftarrow \underline{sasst}\overline{st} \Leftarrow sass\underline{sts}$$

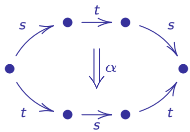
A SRS is a 2-polygraph!

- X can be interpreted as an oriented graph with one vertex $\bullet$, and $x : \bullet \to \bullet$ for any $x \in X$.
- X^* is the free 1-category on this oriented graph.
- R is a cellular extension of X^*, that is a set equipped with source and target maps $s_1, t_1 : R \to X^*$.



with globular relations: $s_0 s_1 = s_0 t_1$, $t_0 s_1 = t_0 t_1$.

- **Example :** $\langle\, \bullet,\ \{s, t\},\ \{sts \overset{\alpha}{\Rrightarrow} tst\}\,\rangle$



- From generating rules of R, we define rewriting steps as $C[f] : C[s_1(f)] \Rightarrow C[t_2(f)]$ where C is a context of X^*, given by composing on the left and right with elements of X^*: