Abstract rewriting

Séminaire de réécriture algébrique

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Historical context

- Broadly speaking, rewriting theory is a constructive, directed approach to the study of equivalences.
  - Origins in combinatorial algebra.
  - In 1914, Thue considered transformation rules on combinatorial objects such as graphs, strings, ...
  - The word problem was the first main question in rewriting theory:

Question

Given two objects, can one be transformed into the other via a (finite) application of the transformation rules?

  - The decidability of this question was only resolved in 1947 by Post and Markov independently.
- Rewriting has since found a variety of applications:
  - Theoretical computer science:
    - proof theory, language theory, programming, ...
  - Algebra
    - commutative algebra, homotopical and homological algebra, Lie algebras, higher categories, ...

Abstract rewriting

Many of the central ideas of rewriting theory can be expressed in the setting of **abstract rewriting**:

- We have two ingredients:
  - a set X of objects.
  - a binary relation $R \subseteq X \times X$.
- Such data represents an **abstract rewriting system**.
- In this mini-course we will
  - recall basic terminology from abstract rewriting,
  - formalise the word problem in this context,
  - recall unicity, *i.e.* confluence, properties and their equivalences,
  - recall reachability, *i.e.* termination, and its role.
- We will present both geometric and algebraic interpretations.
- Afterwards, Benjamin will go on to discuss string rewriting systems.

Abstract rewriting systems

So what is an **abstract rewriting system (ARS)**?

- Consists of a set X and

$$\text{a } \textbf{rewrite relation} \rightarrow \subseteq X \times X$$

  - for $(x, y) \in \rightarrow$, we write $x \rightarrow y$; y is a **one-step reduct** of x.
  - the **converse relation** is denoted by $\leftarrow$.
- We consider **reduction sequences**, *i.e.* equalities or finite sequences of steps:

$$x \equiv x \qquad \text{or} \qquad x \equiv x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_{n-1} \rightarrow x_n \equiv y$$

  and say that x **reduces** to y, denoted by $x \xrightarrow{*} y$.
- We also consider **zigzag sequences**, *i.e.*

$$x \equiv x_0 \xleftarrow{*} x_1 \xrightarrow{*} \cdots \xleftarrow{*} x_{n-1} \xrightarrow{*} x_n \equiv y$$

  and say that x is **equivalent** to y, denoted by $x \xleftrightarrow{*} y$.
- Goal: capture the equivalence relation via the rewrite relation.

Terminology

- The **composition** of relations is defined by

$$x \to_a \cdot \to_b z \iff \exists y \text{ such that } x \to_a y \text{ and } y \to_b z,$$

the **identity relation** being denoted by $\to^0 := \{(x,x)\ ,\ x \in X\}$.

- We define $\to^n := \to \cdot \to^{n-1}$ for any $n \geq 1$.
- The **transitive closure** of $\to$ is defined by $\xrightarrow{+} := \cup_{n \geq 1} \to^n$.
- The **reflexive, transitive closure** of $\to$ is defined by

$$\xrightarrow{*} := \xrightarrow{+} \cup \to^0 = \bigcup_{n \geqslant 0} \longrightarrow^n$$

and thus contains all reduction sequences of $\to$.

- the **symmetric, reflexive, transitive closure** of $\to$ is defined by

$$\xleftrightarrow{*} := (\longleftrightarrow)^* = (\leftarrow \cup \to)^*$$

and thus contains all of zigzag sequences of $\to$.

Abstract rewriting systems

- These notions can be interpreted in (at least) two ways:
    - **Algebraically**: relation algebras (Kleene algebras)

$$\underbrace{(\mathcal{P}(X \times X),\ \cdot,\ \to^0,\ \cup,\ \emptyset}_{\text{semi-anneau,}},\ (-)^*)$$

    - **Geometrically**: directed graphs (1-polygraphs)

$(X, \to)$

$$X = V \overset{t}{\underset{s}{\leftleftarrows}} E = \leftarrow \subseteq X \times X$$

- The **word problem** in the context of an ARS is stated as follows:

Question

Given x and y in X, do we have $x \overset{*}{\longleftrightarrow} y$?

- We will see that relations $\to$ which are confluent and terminating admit decidable word problems.

Example: $\mathbb{N}^2$

- Consider the set X of free words on the alphabet $\{x, y\}$.
- Let $\to$ be the binary relation on X defined by

$$uyxv \to uxyv \qquad \forall u, v \in X.$$

- The equivalence generated by $\to$ is such that $X / \xleftrightarrow{*} \cong \mathbb{N}^2$:

$$[w] \longmapsto (n, m),$$

  where n (resp. m) is the number of occurrences of x (resp. y) in w.
- A priori, we must look at all zigzag sequences to understand the quotient...
- Reduction sequences move occurrences of x to the left and occurrences of y to the right.
- This directedness will allow us to proceed as follows:
  - Existence: reduce each $w \in X$ to a terminal element $\hat{w}$.
  - Unicity: show that the element $\hat{w}$ is unique.

Branchings and confluences

a.k.a. disagreements and agreements…

- A **local branching** of $\to$ is an element of $\leftarrow \cdot \to$

$$\begin{tikzcd} & \cdot \arrow[dl] \arrow[dr] & \\ x & & y \end{tikzcd} \qquad (x, y) \ \in \leftarrow \cdot \to$$

- A **branching** of $\to$ is an element of $\overset{*}{\longleftarrow} \cdot \overset{*}{\longrightarrow}$:

$$\begin{tikzcd} & \cdot \arrow[dl, "*"'] \arrow[dr, "*"] & \\ x & & y \end{tikzcd} \qquad (x, y) \ \in \overset{*}{\longleftarrow} \cdot \overset{*}{\longrightarrow}$$

- A **confluence** of $\to$ is an element of $\overset{*}{\longrightarrow} \cdot \overset{*}{\longleftarrow}$

$$\begin{tikzcd} x \arrow[dr, "*"'] & & y \arrow[dl, "*"] \\ & \cdot & \end{tikzcd} \qquad (x, y) \ \in \overset{*}{\longrightarrow} \cdot \overset{*}{\longleftarrow}$$

Confluence properties : unicity

We say that $\rightarrow$ is …

- **locally confluent** if every local branching is confluent, *i.e.*

$$
\begin{tikzcd}
 & \cdot \arrow[dl] \arrow[dr] & \\
x \arrow[dr, dashed, "*"'] & & y \arrow[dl, dashed, "*"] \\
 & \cdot &
\end{tikzcd}
\qquad\qquad
\leftarrow \cdot \rightarrow \;\subseteq\; \xrightarrow{*} \cdot \xleftarrow{*}
$$

Confluence properties : unicity

We say that $\rightarrow$ is ...

- (globally) **confluent** if every branching is confluent, *i.e.*

$$\begin{tikzcd} & \cdot \arrow[dl, "*"'] \arrow[dr, "*"] & \\ x \arrow[dr, dashed, "*"'] & & y \arrow[dl, dashed, "*"] \\ & \cdot & \end{tikzcd}$$

$$\xleftarrow{*} \cdot \xrightarrow{*} \subseteq \xrightarrow{*} \cdot \xleftarrow{*}$$

Confluence properties : unicity

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$$\begin{tikzcd} & \cdot \arrow[dl, "*"'] \arrow[dr, "*"] & \\ x \arrow[dr, dashed, "*"'] & & y \arrow[dl, dashed, "*"] \\ & \cdot & \end{tikzcd}$$

$$\xleftarrow{*} \cdot \xrightarrow{*} \subseteq \xrightarrow{*} \cdot \xleftarrow{*}$$

- **Church-Rosser** if every zigzag is confluent, *i.e.*

$$\begin{tikzcd} x \equiv x_0 \arrow[r, leftrightarrow] \arrow[rr, bend left, leftrightarrow, "*"] \arrow[dr, dashed, "*"'] & \cdots \arrow[r, leftrightarrow] & x_n \equiv y \arrow[dl, dashed, "*"] \\ & \cdot & \end{tikzcd}$$

$$\xleftrightarrow{*} \subseteq \xrightarrow{*} \cdot \xleftarrow{*}$$

Confluence properties : unicity

We say that $\rightarrow$ is …

- (globally) **confluent** if every branching is confluent, *i.e.*

∀

$$\begin{tikzcd} & \cdot \arrow[dl, "*"'] \arrow[dr, "*"] & \\ x \arrow[dr, dashed, "*"'] & & y \arrow[dl, dashed, "*"] \\ & \cdot & \end{tikzcd}$$

∃

$$\xleftarrow{*} \cdot \xrightarrow{*} \subseteq \xrightarrow{*} \cdot \xleftarrow{*}$$

- **Church-Rosser** if every zigzag is confluent, *i.e.*

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$$\xleftrightarrow{*} \subseteq \xrightarrow{*} \cdot \xleftarrow{*}$$

- Existential quantification on the confluences is given by the inclusion $\subseteq$.
- These properties express coherence of the ARS.

Coherence via branchings

Theorem (Churh-Rosser)

Let $(X, \to)$ be an abstract rewriting system. Then

$$\to \text{ is confluent} \quad \Longleftrightarrow \quad \to \text{ is Church-Rosser.}$$

The proof is by induction on the length l of a zigzag sequence:

$(\Leftarrow)$ trivial : $\xleftarrow{*} \cdot \xrightarrow{*} \subseteq \xleftrightarrow{*} \subseteq \xrightarrow{*} \cdot \xleftarrow{*}$

$(\Rightarrow)$ $x \xleftrightarrow{*} y$: if $l = 0$ $x \equiv y$ so confluent.

if $l > 0$

$$\begin{tikzcd} x \arrow[r, leftrightarrow, "*"] \arrow[dr, dashed, "*"'] & y' \arrow[r] \arrow[d, dashed, "*"] \arrow[dl, phantom, "\mathrm{IH.}"] \arrow[dr, phantom, "\mathrm{Conf.}"] & y \arrow[d, dashed, "*"] \\ & z \arrow[r, dashed, "*"] & z' \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} x \arrow[r, leftrightarrow, "*"] \arrow[dr, dashed, "*"'] & y' \arrow[d, dashed, "*"] \arrow[dl, phantom, "\mathrm{IH.}"] \arrow[dr, phantom, "="] & y \arrow[l] \arrow[dl, dashed, "*"] \\ & z & \end{tikzcd}$$

$\square$

- We have reduced the problem from zigzags to branchings.

Normalization and termination : existence

- An element $x \in X$ is a **normal form**, or **irreducible**, for $\to$ if

$$\forall y \in X, \qquad \neg(x \to y).$$

- We say that $\to$ is **normalising** if

$$\forall x \in X, \exists\, x' \quad \text{such that} \quad x \xrightarrow{*} x' \quad \text{and } x' \text{ is a normal form.}$$

- We say that $\to$ is **terminating**, or **Noetherian**, if all reduction sequences are of finite length.
- For every $A \subseteq X$, let $\Diamond_{\to}(A) := \{x \in X \mid \exists a \in A \text{ s.t. } x \to a\}$.
- An algebraic characterisation of termination is the following:

$$\forall A \subseteq X, \qquad A \subseteq \Diamond_{\to}(A) \quad \Rightarrow \quad A = \emptyset$$

termination of $\to$ $\Leftrightarrow$ $\forall A \subseteq X$, $A \neq \emptyset$, A has a ~~maximal element~~ normal form for $\to$.

Normalization and termination : existence

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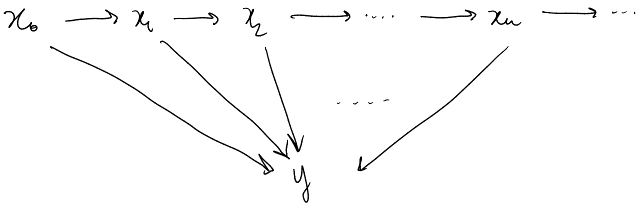
- We say that $\to$ is **terminating**, or **Noetherian**, if all reduction sequences are of finite length.
- Termination implies normalisation, but the converse does not hold.



Noetherian induction

The importance of being terminating...

- Let $(X, \to)$ an ARS and $\mathcal{P}$ a property on elements of X.
- The principle of **Noetherian induction** can be stated as follows: if

$$\forall x \in X, \Big[\Big(\forall y \in X, \quad x \xrightarrow{+} y \Rightarrow \mathcal{P}(y) \Big) \Rightarrow \mathcal{P}(x) \Big],$$

  then $\mathcal{P}(x)$ holds for all $x \in X$.
- We can use this principle when dealing with terminating ARS's:

Proposition

If $\to$ is terminating if, and only if, the principle of Noetherian induction holds.

- $\to$ is **convergent** when it is both terminating and confluent.
- In that case, every $x \in X$ admits a unique normal form which will be denoted by $\hat{x}$.

From local to global

Theorem (Newman's lemma)

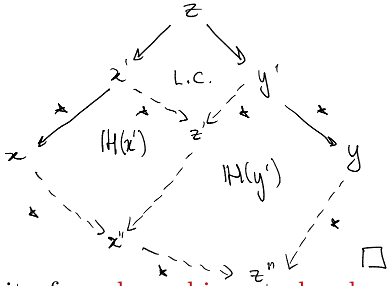
Let $(X, \to)$ be a ~~convergent~~ terminating abstract rewriting system. Then

$$\to \text{ is locally confluent} \iff \to \text{ is confluent}$$

The proof is by Noetherian induction: ($\Leftarrow$) trivial

($\Rightarrow$) - If z is a normal form, $x \equiv z \equiv y$, so confluent.

- If z is not a normal form, $\exists\, x', y'$ as in the diagram.



$\square$

- We have reduced the problem of unicity from branchings to local branchings.

Convergence and the word problem

- Now that we have seen the fundamental definitions and results of abstract rewriting, let's get back to the word problem:

Question

Given x and y in X, do we have $x \overset{*}{\longleftrightarrow} y$?

- If $(X, \rightarrow)$ is a normalising and confluent ARS, then

$$x \overset{*}{\longleftrightarrow} y \qquad \Longleftrightarrow \qquad \hat{x} \equiv \hat{y}.$$

- So if the normal forms are computable and the identity $\equiv$ on X is decidable, so is the word problem!
- When $\rightarrow$ terminates (and satisfies a finiteness condition...) we can compute normal forms. This gives the following result:

Proposition

The word problem associated to a convergent ARS is decidable.

Presentation of $\mathbb{N}^2$

$uyxv \longrightarrow uxyv$ $\quad \underbrace{\text{termine}}_{\substack{\text{ordre lexico} \\ x<y}}$ et est confluent.

$$
\begin{tikzcd}
w \arrow[rr, leftrightarrow, "*", "?"'] \arrow[d, dashed] & & w' \arrow[d, dashed] \\
x^{n} y^{m} & = & x^{n'} y^{m'}
\end{tikzcd}
$$

Time for strings!