

Linear polygraphs.

Benjamin Dupont

Institut Camille Jordan, Université Lyon 1

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 - ▶ Orientation of relations depend on an ambient monomial order, that is a well-founded total order such that $s(f) > t(f)$ for any rule f and $uvw > uv'w$ for any monomials u, v, w such that $v > v'$.

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- ▶ Introduction of **linear polygraphs**:

CONVERGENT PRESENTATIONS AND POLYGRAPHIC RESOLUTIONS OF ASSOCIATIVE ALGEBRAS

YVES GUIRAUD

ERIC HOFFBECK

PHILIPPE MALBOS

Abstract – Several constructive homological methods based on noncommutative Gröbner bases are known to compute free resolutions of associative algebras. In particular, these methods relate the Koszul property for an associative algebra to the existence of a quadratic Gröbner basis of its ideal of relations. In this article, using a higher-dimensional rewriting theory approach, we give several improvements of these methods. We define polygraphs for associative algebras as higher-dimensional linear rewriting systems that generalise the notion of noncommutative Gröbner bases, and allow more possibilities of termination orders than those associated to monomial orders. We introduce polygraphic resolutions of associative algebras, giving a categorical description of higher-dimensional syzygies for presentations of algebras. We show how to compute polygraphic resolutions starting from a convergent presentation, and how these resolutions can be linked with the Koszul property.

Keywords – Higher-dimensional associative algebras, confluence and termination, linear rewriting, polygraphs, free resolutions, Koszulness.

M.S.C. 2010 – **Primary**: 18G10, 16Z05. **Secondary**: 68Q42, 18D05.

A toy example

- ▶ Consider an associative algebra $A = \mathbb{K}\langle x, y, z \mid xyz - x^3 - y^3 - z^3 \rangle$, i.e. A is the algebra generated by x, y and z quotiented by the ideal generated by $xyz - x^3 - y^3 - z^3$.

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- ▶ If we orient the relation as $xyz \Rightarrow x^3 + y^3 + z^3$, this **can not** be compatible with a monomial order.
 - ▶ Suppose such an order \prec exists.
 - ▶ Since \prec is total, one of x, y, z is greater than the other two. Suppose it is x .
 - ▶ Then $x \succ y$ implies $x^2 \succ yx$ and $x \succ z$ implies $yx \succ yz$.
 - ▶ Hence $x^2 \succ yz$, and $x^3 \succ xyz$.
 - ▶ We show that any x^3, y^3 and z^3 is greater than xyz .

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- ▶ Consider the map $\Phi : \{x, y, z\}^* \rightarrow \mathbb{N}$ defined by

$$\Phi(u) := 3 \times \text{number of } xyz \text{ in } u + \text{number of } y \text{ in } u.$$

- ▶ $\Phi(uxyzv) > \Phi(ux^3v)$, $\Phi(uxyzv) > \Phi(uy^3v)$, $\Phi(uxyzv) > \Phi(uz^3v)$ for any $u, v \in \{x, y, z\}^*$.

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- ▶ Therefore, if we have a rewriting step $f \Rightarrow \sum_i \lambda_i f_i$, we have $\Phi(f) > \Phi(f_i)$.
- ▶ There cannot exist an infinite rewriting sequence $f \Rightarrow f' \Rightarrow f'' \Rightarrow \dots$ in P , otherwise there would be a strictly decreasing infinite sequence of natural numbers

$$\Phi(f) > \Phi(f') > \Phi(f'') > \dots$$

I. Linear 2-polygraphs

II. The linear critical branching theorem

III. Squier's coherence theorem

IV. Higher-dimensional linear polygraphs

I. Linear 2-polygraphs

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Monoids \leftrightarrow 1-categories with only one 0-cell.

Associative algebras \leftrightarrow 1-algebroids with only one 0-cell.

- ▶ A 1-algebroid over a field \mathbb{K} is a 1-category enriched over the category $\mathbf{Vect}_{\mathbb{K}}$ of \mathbb{K} -vector spaces.
- ▶ Explicitly, it is given by:
 - ▶ a set of 0-cells A_0 ,
 - ▶ for every 0-cells p and q , a \mathbb{K} -vector space $A(p, q)$, whose elements are the 1-cells of A .
 - ▶ for any 0-cells p, q and r , there is a \mathbb{K} -linear map $\star_0 : A(p, q) \otimes A(q, r) \rightarrow A(p, r)$, and we denote $\star_0(f \otimes g)$ by fg .
 - ▶ this composition is associative: $(fg)h = f(gh)$, and unitary: $1_p f = f = f 1_q$ for any $f \in A(p, q)$.

- ▶ Let (P_0, P_1) be a 1-polygraph, i.e., a directed graph with source and target maps s_0, t_0 .
- ▶ The **free 1-algebroid** over P is the 1-algebroid P_1^ℓ defined by:
 - ▶ $(P_1)_0^\ell = P_0$,
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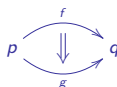
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- ▶ A **linear 2-polygraph** is a triple (P_0, P_1, P_2) made of
 - ▶ a 1-polygraph (P_0, P_1) ,
 - ▶ a cellular extension P_2 of the free 1-algebroid P_1^ℓ , with source and target maps s_1, t_1 satisfying globular relations:

$$s_0 s_1 = s_0 t_1, \quad y_0 t_1 = t_0 t_1.$$

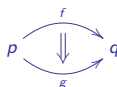
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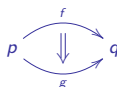


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- ▶ **From now on, we consider linear 2-polygraphs with only one 0-cell.**
- ▶ The **ideal** of a linear 2-polygraph P is the two-sided ideal of the algebra P_1^ℓ generated by

$$\{s_1(\alpha) - t_1(\alpha) \mid \alpha \in P_2\}$$

The **algebra presented by P** is the \mathbb{K} -algebra given by $P_1^\ell / I(P)$.

2-Algebras

- ▶ A **2-algebra** is an internal category in the category $\mathbf{Alg}_{\mathbb{K}}$ of associative \mathbb{K} -algebras.
- ▶ Explicitly, it is given by a diagram in \mathbf{Alg} :

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- ▶ The product of two 2-cells a and b in A_2 is denoted by ab .
- ▶ The linear structure and product in the algebra $A_2 \times_{A_1} A_2$ are given by:

$$(a, a') + (b, b') = (a + b, a' + b'), \quad \lambda(a, a') = (\lambda a, \lambda a'), \quad (a, a')(b, b') = (ab, a' b')$$

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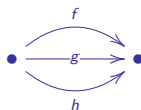
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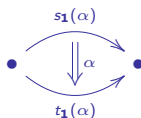
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- ▶ The morphisms s_1 , t_1 and \star_1 satisfy the axioms of a 1-category.
- ▶ Elements of A_1 are called **1-cells of A** , and are pictured as:



- ▶ Elements of A_2 are called **2-cells of A** , and are pictured as



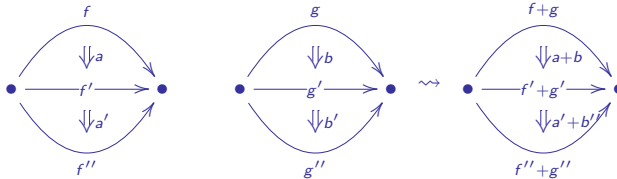
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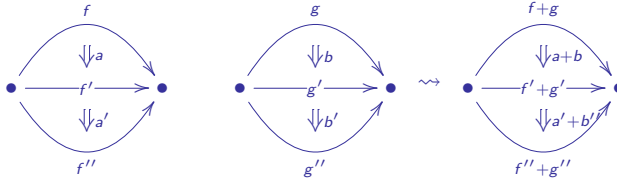
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- ▶ **Some identities:**

- ▶ For any 1-composable a and a' , we have $a \star_1 a' = a + a' - t_1(a)$,
- ▶ For any 2-cell a , we have $a^- = -a + s_1(a) + t_1(a)$, so that any 2-cell is invertible for the \star_1 -composition.
- ▶ For any 2-cell a and b in A , we have

$$ab = as_1(b) + t_1(a)b - t_1(a)s_1(b) \underset{\text{Eq}_{a,b}}{=} s_1(a)b + at_1(b) - s_1(a)t_1(b)$$

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i_2 (curved arrow from P_2^ℓ to P_1^ℓ)
 s_1 (top straight arrow from P_1^ℓ to P_2^ℓ)
 t_1 (bottom straight arrow from P_1^ℓ to P_2^ℓ)
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- It has a structure of algebra, with product defined by

$$ab = as_1(b) + t_1(a)b - t_1(a)s_1(b)$$

- Elements of P_2^ℓ have shape

$$\bullet \xrightarrow{f} \bullet \begin{array}{c} \xrightarrow{\lambda s_1(\alpha)} \\ \Downarrow \lambda \alpha \\ \xrightarrow{\lambda t_1(\alpha)} \end{array} \bullet \xrightarrow{g} \bullet + \bullet \xrightarrow{h} \bullet$$

The free 2-algebra on a linear 2-polygraph

- Given a linear 2-polygraph P , the **free 2-algebra over P** is the 2-algebra given by the following diagram

$$\begin{array}{c}
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 \xrightarrow{i_2} \\
 \text{---} \xrightarrow{s_1} \text{---} \\
 \xleftarrow{t_1}
 \end{array}
 P_1^\ell \leftarrow P_2^\ell \xleftarrow{*_1} P_2^\ell \times_{P_1^\ell} P_2^\ell
 \end{array}$$

where:

- P_2^ℓ is defined as the quotient of the P_1^ℓ -bimodule $(P_1^\ell \otimes \mathbb{K}P_2 \otimes P_1^\ell) \oplus P_1^\ell$ quotiented by the equivalence relation generated by

$$\{Eq_{a,b} \mid a, b \in P_1^\ell \otimes \mathbb{K}P_2 \otimes P_1^\ell\}.$$

- It has a structure of algebra, with product defined by

$$ab = as_1(b) + t_1(a)b - t_1(a)s_1(b)$$

- Elements of P_2^ℓ have shape

$$\bullet \xrightarrow{f} \bullet \begin{array}{c} \xrightarrow{\lambda s_1(\alpha)} \\ \Downarrow \lambda \alpha \\ \xleftarrow{\lambda t_1(\alpha)} \end{array} \bullet \xrightarrow{g} \bullet + \bullet \xrightarrow{h} \bullet$$

- The sphere correspond to the "monomial" place where we will apply rewriting steps inside a polynomial.

Monomials and rewriting steps

- ▶ A **monomial** in P_2^ℓ is a 1-cell in the free monoid P_1^* over P_1 .
 - ▶ The monomials of P_2^ℓ form a linear basis of the algebra P_1^ℓ .
 - ▶ Every 1-cell $f \neq 0$ of P_1^ℓ can be uniquely written as $f = \lambda_1 u_1 + \dots + \lambda_p u_p$ with $\lambda_i \in \mathbb{K} \setminus \{0\}$ and u_i monomials.

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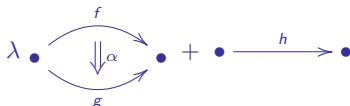
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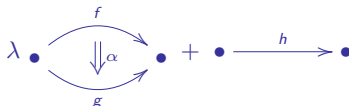
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- ▶ **Why the green condition ?** To avoid termination obstructions: if $f \Rightarrow g$ is a rewriting step, then $-f \Rightarrow -g$, and thus

$$g = (f + g) - f \Rightarrow (f + g) - g = f.$$

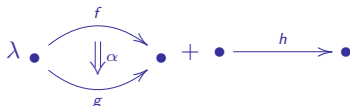
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- ▶ A 2-cell of P_2^ℓ with that shape but without the green condition is called **elementary**.

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 - ▶ We have $x^2 \prec_P xy$ and $y^2 \prec_P xy \rightsquigarrow x^2y \succ_P xy^2 \succ_P x^2y$.
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- ▶ Denote P_1^{nf} the set of normal forms of P . If P is terminating,

$$P_1^\ell = P_1^{\text{nf}} + I(P), \quad f = \widehat{f} + (f - \widehat{f}).$$

- ▶ If $f = f_0 \Rightarrow f_1 \Rightarrow f_2 \Rightarrow \dots \Rightarrow f_n = \widehat{f}$, then

$$f - \widehat{f} = (f_0 - f_1) + (f_1 - f_2) + \dots + (f_{n-2} - f_{n-1}) + (f_{n-1} - f_n) \in I(P).$$

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► The decomposition is not direct in general: consider $P = \langle x, y \mid x^2 \stackrel{\beta}{\Rightarrow} xy \rangle$.

$$\begin{array}{c}
 \xrightarrow{\beta x} xyx \\
 x^3 \\
 \xrightarrow{x\beta} x^2y \xrightarrow{\beta y} xy^2
 \end{array}$$

$$xyx - xy^2 = -(x^2 - xy)x + x(x^2 - xy) + (x^2 - xy)y$$

► **Theorem:** The following conditions are equivalent:

- i) P is confluent.
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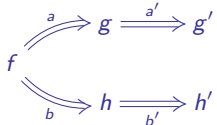
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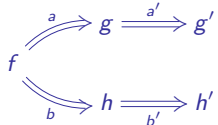
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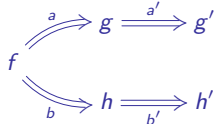
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► **Theorem:** Let A be an algebra and P be a convergent linear 2-polygraph presenting A . The set P_1^{mnf} of monomials of P_1^ℓ in normal form w.r.t P is a **linear basis** of A .

II. The linear critical branching theorem

- ▶ Associative algebras over a field \mathbb{K} are presented by **linear 2-polygraphs**. These are triples $P = (P_0, P_1, P_2)$ where:

- ▶ $P_0 = \{\bullet\}$,

- ▶ generating 1-cells in $P_1 \ni x, y, z, \dots \rightsquigarrow P_1^\ell \ni xyz - x^3 - y^3 - z^3$,

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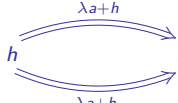
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- ▶ If P is a linear 2-polygraph, $(P_1^\ell, \Rightarrow_{\text{stp}})$ gives an abstract rewriting system.

- ▶ As opposed to set-theoretical context, we do not consider all the 2-cells of P_2^ℓ .

- ▶ **Newman lemma**: If P is terminating, confluence and local confluence are equivalent properties.

- ▶ Local branchings of a left-monomial linear 2-polygraph are split into the following four families:

1) **Aspherical:** $\lambda f + h$  $\lambda g + h$

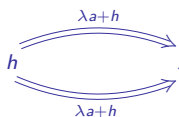
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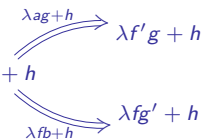
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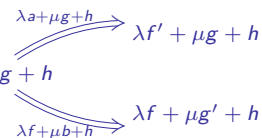
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Local confluence from critical confluence

- ▶ **Example:** Consider the linear 2-polygraph $P = \langle x, y, z, t \mid xy \xrightarrow{\alpha} xz, zt \xrightarrow{\beta} 2yt \rangle$.
 - ▶ It has no critical branching, but it has a non-confluent additive branching:

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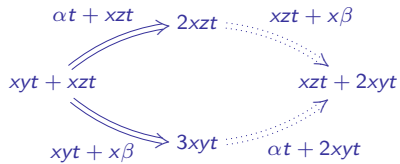
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$$\begin{array}{ccc} & \alpha t + xzt & \rightarrow 2xzt \\ & \nearrow & \\ xyt + xzt & & \\ & \searrow & \\ & xyt + x\beta & \rightarrow 3xyt \end{array}$$

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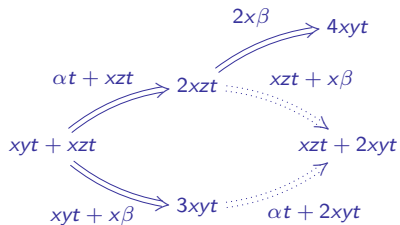
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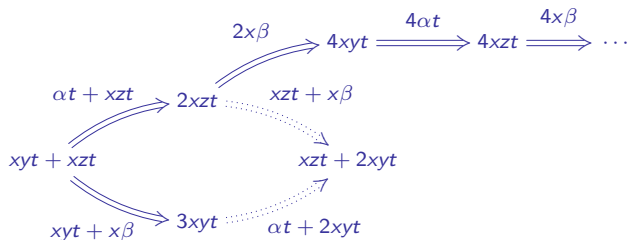
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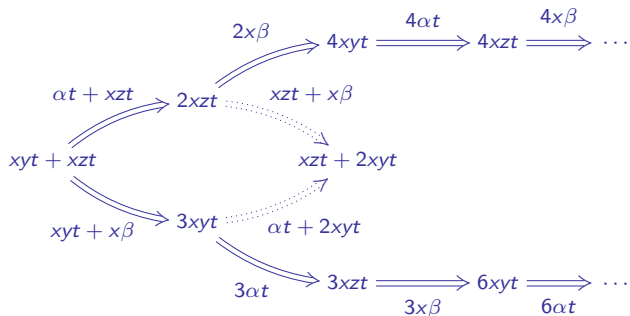
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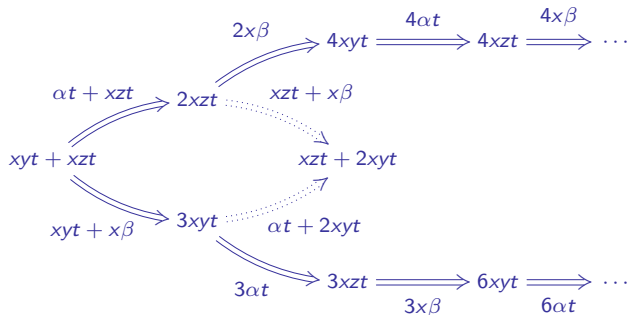
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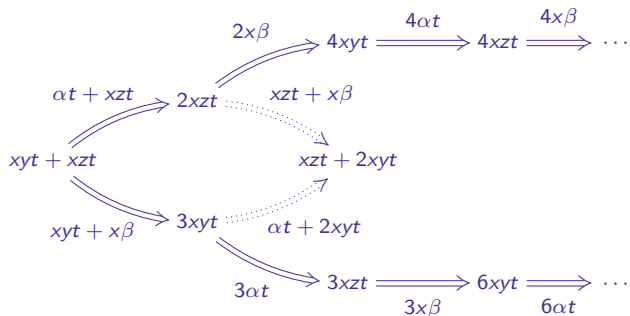


► We need termination to ensure confluence of additive branchings.

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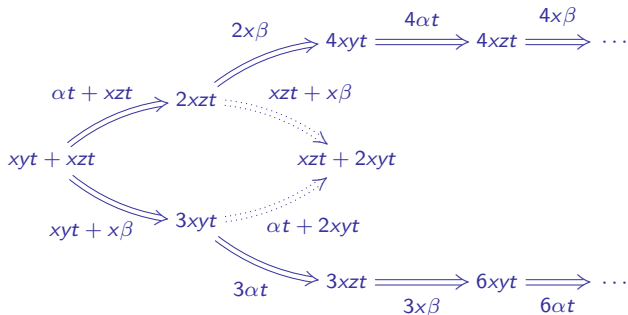
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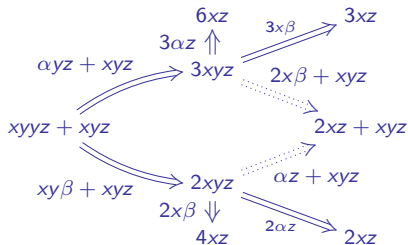
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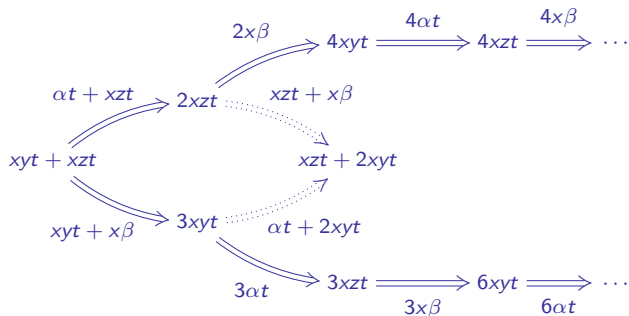
► It terminates, but has a nonconfluent Peiffer branching:



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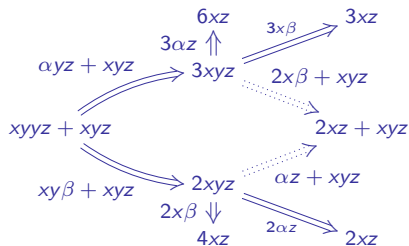
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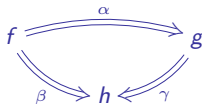
► Critical confluence is needed to ensure confluence of Peiffer branchings.

The linear critical branching theorem

- ▶ **Theorem:** A **terminating** left-monomial linear 2-polygraph is locally confluent if and only if all its critical branchings are confluent.

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$$\begin{array}{ccc}
 & \alpha & \\
 f & \xrightarrow{\quad} & g \\
 & \searrow \quad \swarrow & \\
 & \beta & h & \gamma \\
 & \swarrow \quad \searrow & & \\
 & & &
 \end{array}$$

► **Example:** Let P be a linear 2-polygraph and $\alpha : f \Rightarrow g$ be a 2-cell.

$$\begin{array}{ccc}
 & \lambda\alpha + (\mu f + \eta g) & \\
 \lambda f + (\mu f + \eta g) & \xrightarrow{\quad} & \lambda g + (\mu f + \eta g) \\
 & \searrow \quad \swarrow & \\
 (\lambda + \mu)\alpha + \eta g & \xrightarrow{\quad} & (\lambda + \mu + \eta)g & \xleftarrow{\quad} \mu\alpha + (\lambda + \eta)g
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 \searrow & & \swarrow \\
 & h & \\
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 \end{array}$$

β (left arrow from f to h) γ (right arrow from g to h)

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$$\begin{array}{ccc}
 \lambda f + (\mu f + \eta g) & \xrightarrow{\lambda\alpha + (\mu f + \eta g)} & \lambda g + (\mu f + \eta g) \\
 \searrow & & \swarrow \\
 (\lambda + \mu)\alpha + \eta g & \xrightarrow{\quad} & (\lambda + \mu + \eta)g \\
 \swarrow & & \searrow \\
 & &
 \end{array}$$

- ▶ The proof of the theorem is made by Noetherian induction: consider a local branching with source f , and suppose that any branching (α, β) with source $g \prec_P f$ is confluent.
- ▶ The factorisation property extends to 2-cells of the form $\alpha := f_0 \xrightarrow{\alpha_0} f_1 \xrightarrow{\alpha_1} f_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{p-1}} f_p$ where the α_i are elementary 2-cells.

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- ▶ **Key point:** Let α be an elementary 2-cell of P_2^ℓ , then a can be factorised in the 2-algebra P_2^ℓ into

$$\begin{array}{ccc}
 & \alpha & \\
 f & \xrightarrow{\quad} & g \\
 & \searrow \quad \swarrow & \\
 & \beta & h & \gamma \\
 & \swarrow \quad \searrow & & \\
 & & &
 \end{array}$$

- ▶ **Example:** Let P be a linear 2-polygraph and $\alpha : f \Rightarrow g$ be a 2-cell.

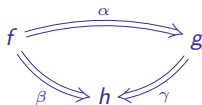
$$\begin{array}{ccc}
 & \lambda\alpha + (\mu f + \eta g) & \\
 \lambda f + (\mu f + \eta g) & \xrightarrow{\quad} & \lambda g + (\mu f + \eta g) \\
 & \searrow \quad \swarrow & \\
 (\lambda + \mu)\alpha + \eta g & \xrightarrow{\quad} & (\lambda + \mu + \eta)g & \xleftarrow{\quad} \mu\alpha + (\lambda + \eta)g
 \end{array}$$

- ▶ The proof of the theorem is made by Noetherian induction: consider a local branching with source f , and suppose that any branching (α, β) with source $g \prec_p f$ is confluent.
- ▶ The factorisation property extends to 2-cells of the form $\alpha := f_0 \xrightarrow{\alpha_0} f_1 \xrightarrow{\alpha_1} f_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{p-1}} f_p$ where the α_i are elementary 2-cells.
 - ▶ By induction on $p \geq 0$. If $p = 0$, α is an identity and the factorisation is trivial.

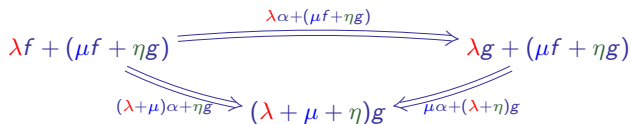
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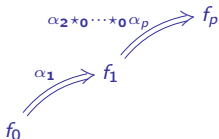


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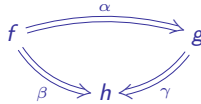
► Otherwise,



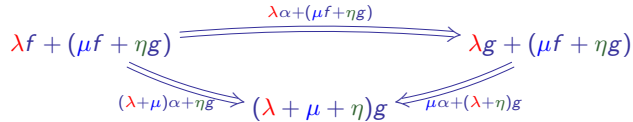
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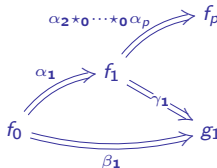


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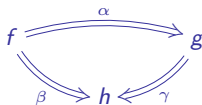
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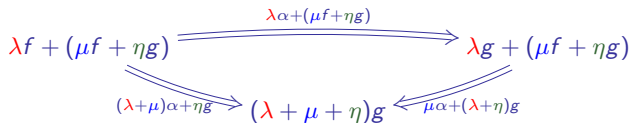
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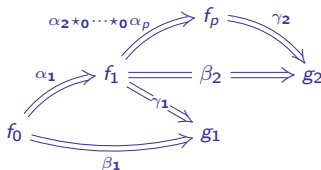
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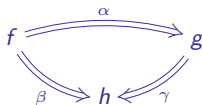
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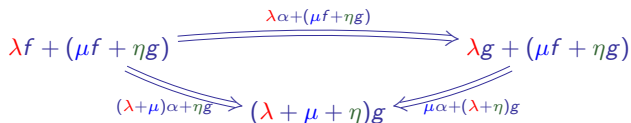
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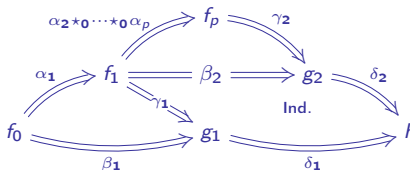


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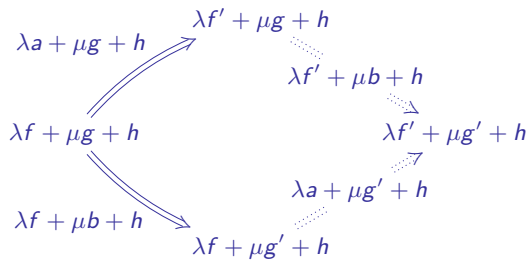
The linear critical branching theorem

- **Step 1:** Additive branchings are confluent.

$$\begin{array}{ccc} & & \lambda f' + \mu g + h \\ & \nearrow & \\ \lambda a + \mu g + h & & \\ & \searrow & \\ \lambda f + \mu g + h & & \\ & \searrow & \\ \lambda f + \mu b + h & & \lambda f + \mu g' + h \end{array}$$

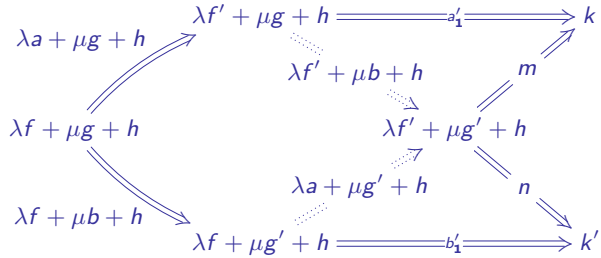
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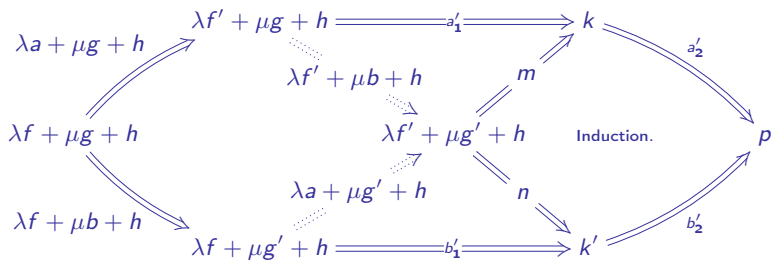
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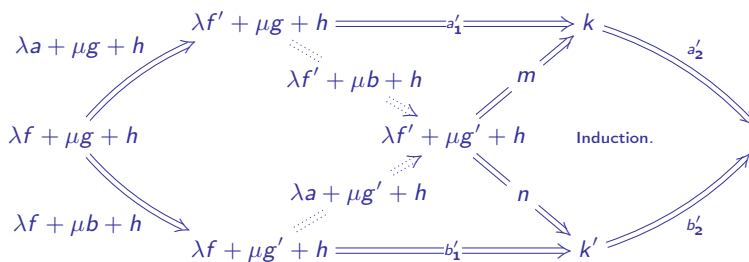
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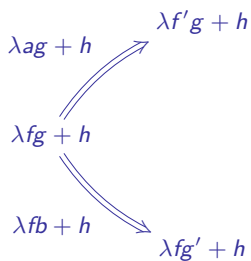


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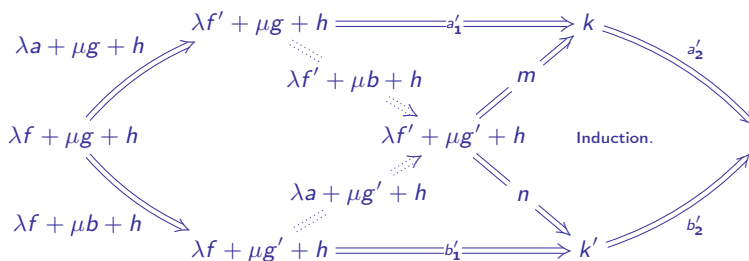


- **Step 2:** Peiffer are confluent.

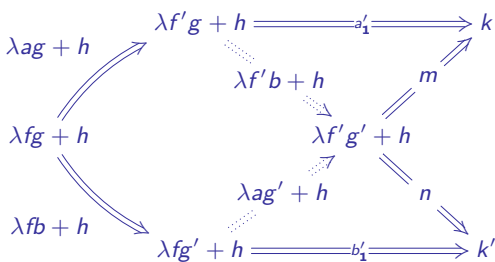


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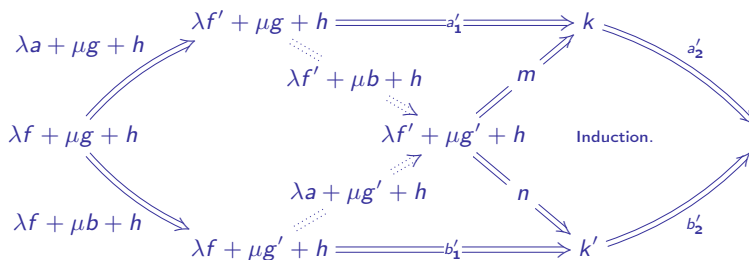


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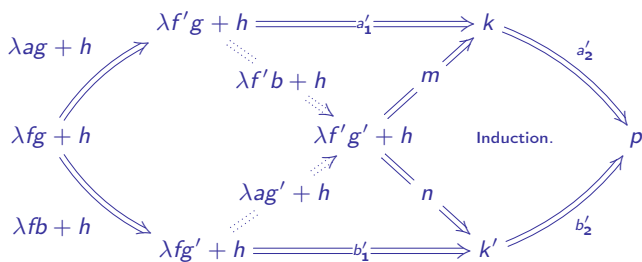


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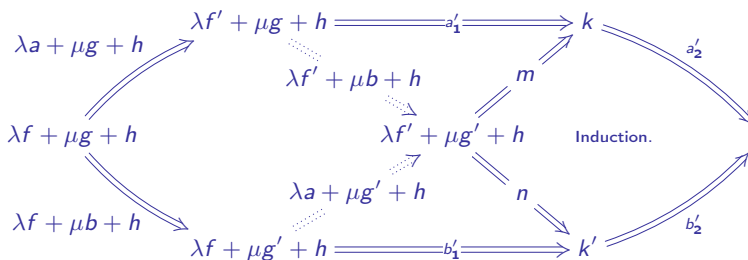


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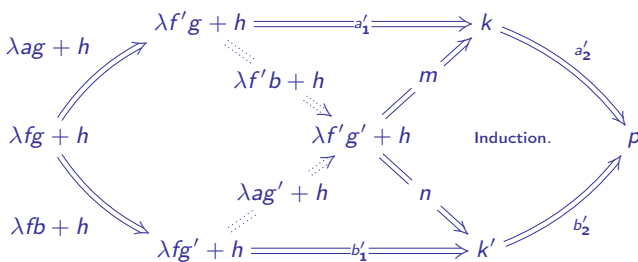


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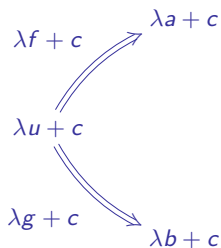
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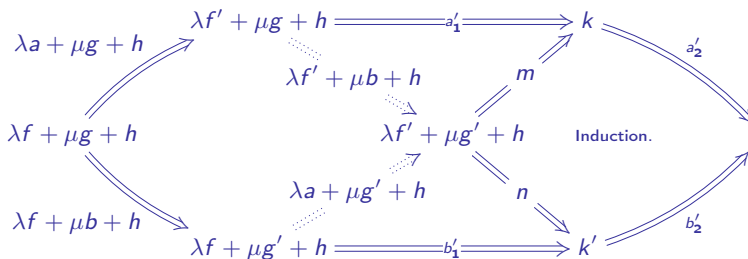


- **Step 3:** Overlappings are confluent.

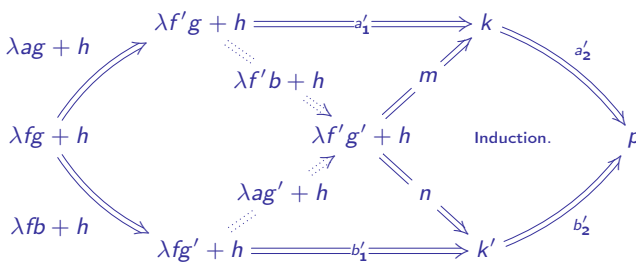


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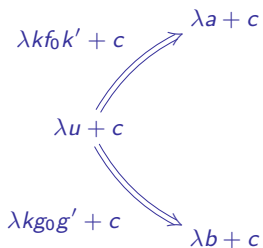
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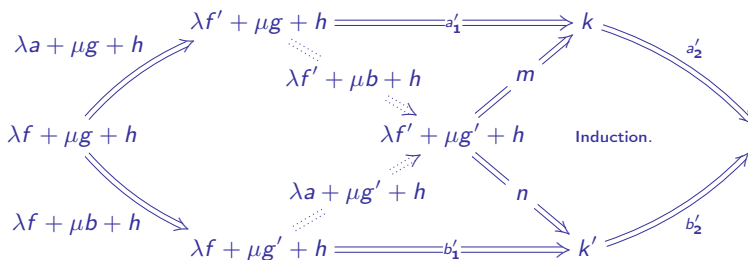


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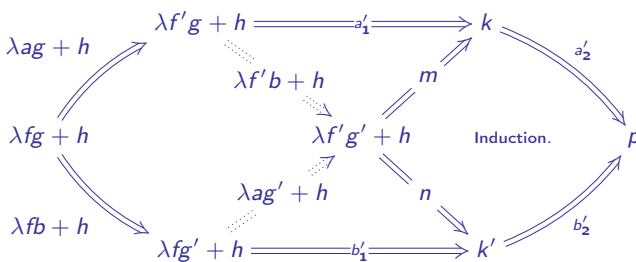


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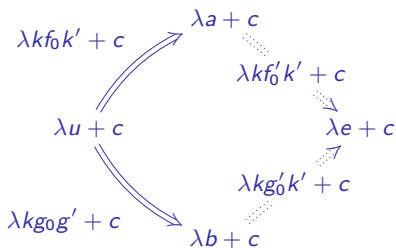
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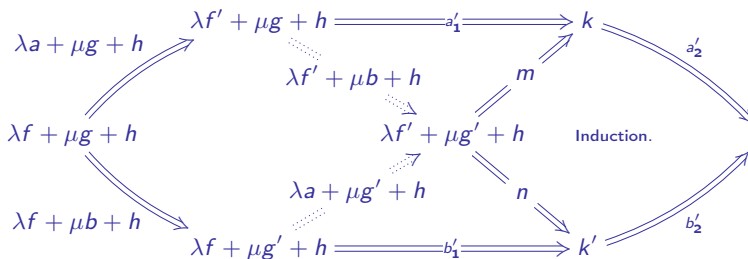


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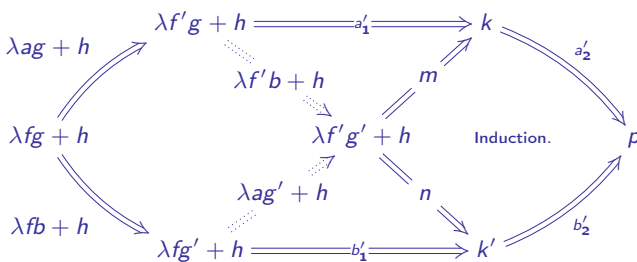


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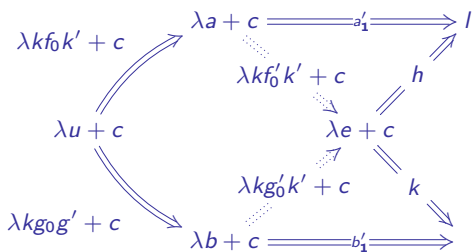
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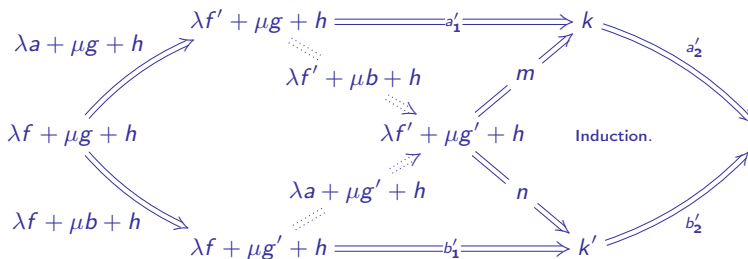


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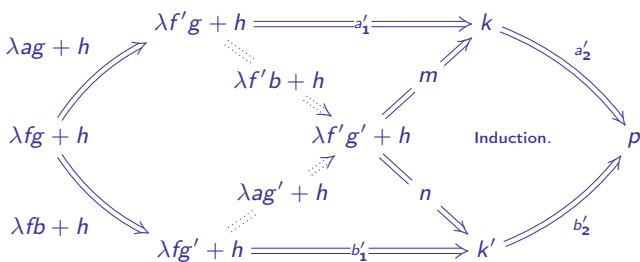


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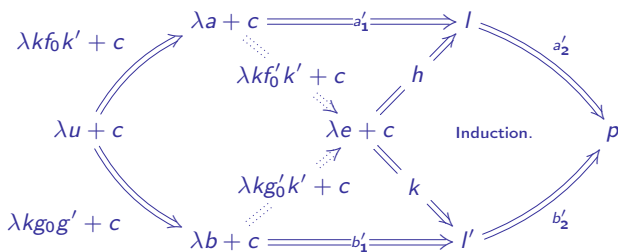
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Example: the Weyl algebras

- ▶ The **Weyl algebra** of dimension n over a field \mathbb{K} is the algebra presented by the linear 2-polygraph

$$P = \langle x_1, \dots, x_n, \partial_1, \dots, \partial_n \mid x_i x_j \Rightarrow x_j x_i, \quad \partial_i \partial_j \Rightarrow \partial_j \partial_i, \quad \partial_i x_j \Rightarrow x_j \partial_i, \\ \partial_i x_i \Rightarrow x_i \partial_i + 1 \text{ for any } 1 \leq i < j \leq n \rangle.$$

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- ▶ It terminates, using the degree lexicographic order on $\partial_1 > \partial_2 > \dots > \partial_n > x_1 > x_2 > \dots > x_n$.

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$$\begin{array}{ccc} & x_i \partial_i x_j + x_j \Rightarrow x_i x_j \partial_i + x_j & \\ \swarrow & & \searrow \\ \partial_i x_i x_j & & x_j x_i \partial_i + x_j \\ \searrow & & \swarrow \\ & \partial_i x_j x_i \Rightarrow x_j \partial_i x_i & \end{array}$$

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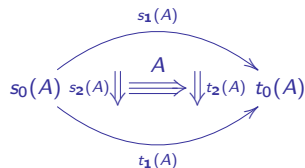
$$\begin{array}{ccc} & \xrightarrow{\quad} & x_i \partial_i x_j + x_j \Rightarrow x_i x_j \partial_i + x_j \\ \partial_i x_i x_j & \xRightarrow{\quad} & \\ & \xrightarrow{\quad} & x_j x_i \partial_i + x_j \\ & \xleftarrow{\quad} & \partial_i x_j x_i \Rightarrow x_j \partial_i x_i \\ & \xleftarrow{\quad} & \end{array}$$

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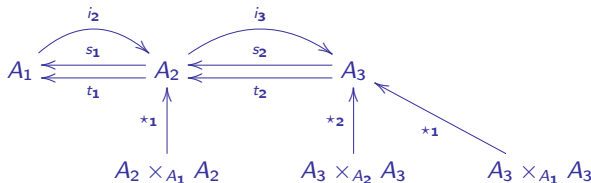
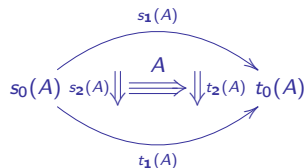
- ▶ A linear basis of the Weyl algebra of dimension n is given by the elements $x_n^{\alpha_n} \dots x_1^{\alpha_1} \partial_n^{\beta_n} \dots \partial_1^{\beta_1}$ for $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \in \mathbb{N}$.

III. Squier's coherence theorem

- ▶ A **linear 3-polygraph** is a quadruple (P_0, P_1, P_2, P_3) made of
 - ▶ a linear 2-polygraph (P_0, P_1, P_2) ,
 - ▶ a cellular extension $P_3 \underset{t_2}{\overset{s_2}{\rightrightarrows}} P_2^\ell$ satisfying globular condition:

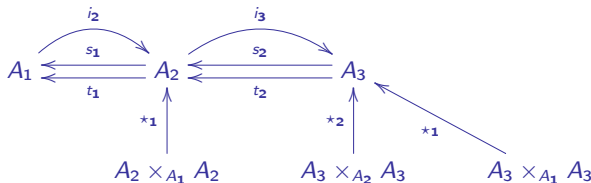
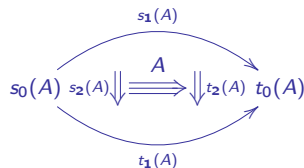


- ▶ A **linear 3-polygraph** is a quadruple (P_0, P_1, P_2, P_3) made of
 - ▶ a linear 2-polygraph (P_0, P_1, P_2) ,
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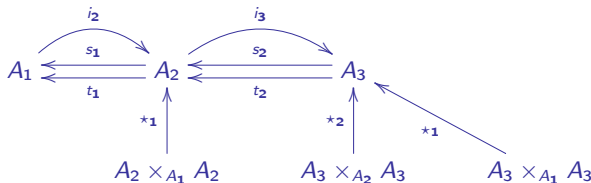
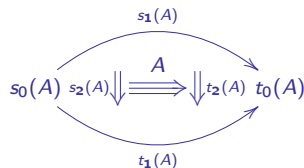
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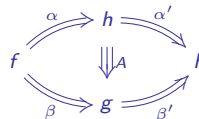


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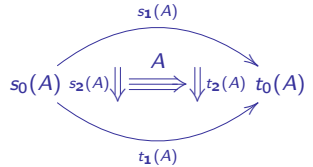
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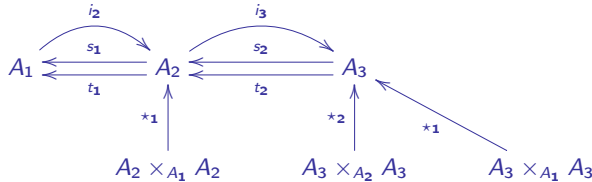
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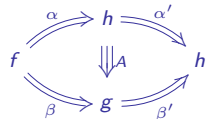
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- ▶ **Theorem (Coherent Newman lemma):** If P is terminating, then P is P_3 -confluent if and only if P is locally P_3 -confluent.

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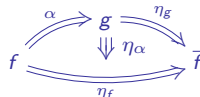
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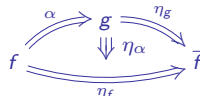
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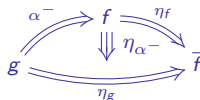
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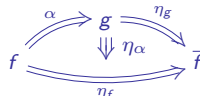
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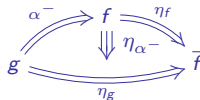
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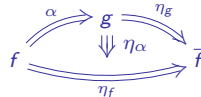
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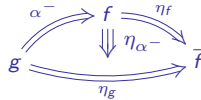
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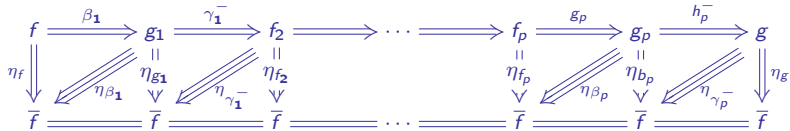


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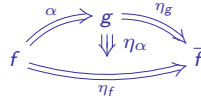
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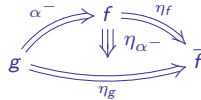
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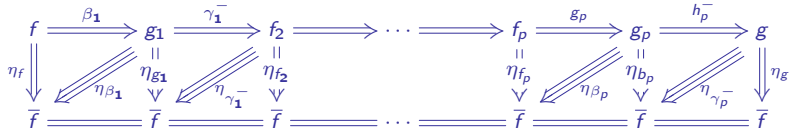
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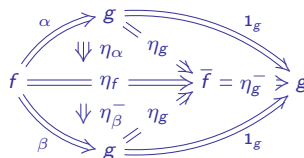
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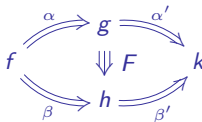
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- Finally, for all parallel 2-cells $\alpha, \beta : f \rightarrow g$ of P_2^ℓ , the composite 3-cell

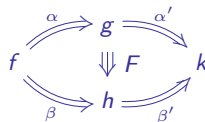


- **Squier's coherence theorem:** Let P be a convergent left-monomial linear 2-polygraph. A cellular extension P_3 of P_2^ℓ that contains a 3-cell



for every critical branching (α, β) of P , with α' and β' positive 2-cells of P_2^ℓ , is acyclic.

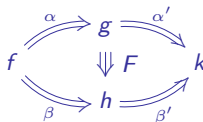
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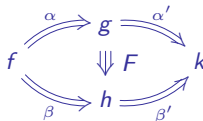


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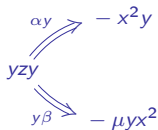
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$$\begin{array}{ccc}
 & \alpha & \\
 & \curvearrowright & \\
 f & \xrightarrow{\quad} & g \\
 & \curvearrowleft & \\
 & \beta & \\
 & \curvearrowright & \\
 & \xrightarrow{\quad} & h \\
 & \curvearrowleft & \\
 & \beta' & \\
 & \curvearrowright & \\
 & \xrightarrow{\quad} & k \\
 & \alpha' & \\
 & \curvearrowright &
 \end{array}
 \quad \Downarrow F$$

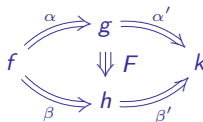
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 & \alpha y & \\
 & \curvearrowright & \\
 yzy & \xrightarrow{\quad} & -x^2 y \\
 & \curvearrowleft & \\
 & y\beta & \\
 & \curvearrowright & \\
 & \xrightarrow{\quad} & -\mu y x^2 \\
 & \curvearrowleft & \\
 & -\mu\gamma & \\
 & \curvearrowright &
 \end{array}$$

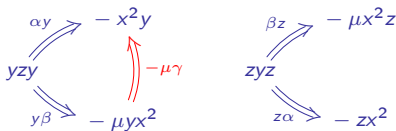
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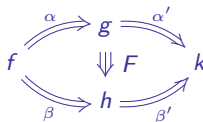
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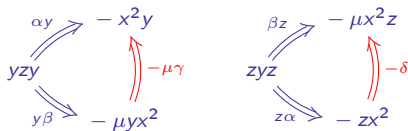


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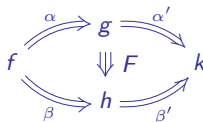
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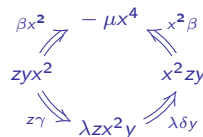
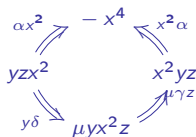
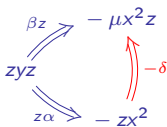
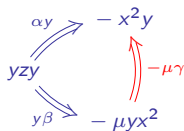
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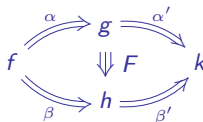
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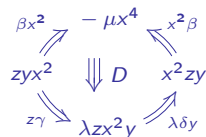
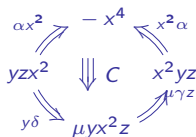
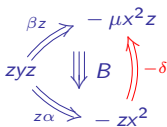
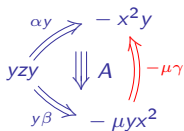
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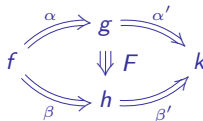
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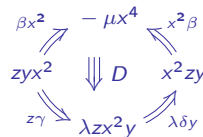
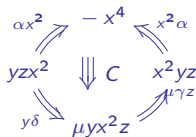
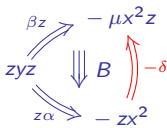
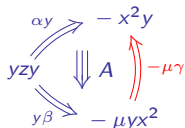


for every critical branching (α, β) of P , with α' and β' positive 2-cells of P_2^ℓ , is acyclic.

- **Example:** Consider $P = \langle x, y, z \mid yz \xrightarrow{\alpha} -x^2, zy \xrightarrow{\beta} -\mu x^2 \rangle$ presenting the algebra $A = \mathbb{K}\langle x, y, z \mid x^2 + yz = 0, x^2 + \lambda zy = 0 \rangle$, with $\mu := \lambda^{-1}$.

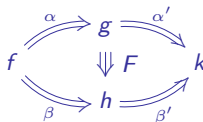
- It terminates, using the deglex order generated by $z > y > x$.

- Squier's completion:



- $\langle x, y, z \mid \alpha, \beta, \gamma, \delta \mid A, B, C, D \rangle$ is a coherent presentation of A .

- **Squier's coherence theorem:** Let P be a convergent left-monomial linear 2-polygraph. A cellular extension P_3 of P_2^ℓ that contains a 3-cell

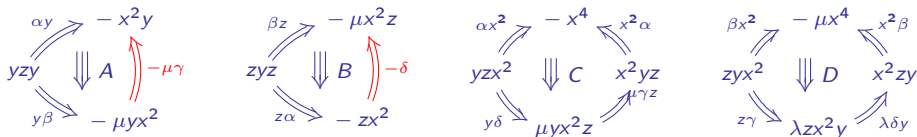


for every critical branching (α, β) of P , with α' and β' positive 2-cells of P_2^ℓ , is acyclic.

- **Example:** Consider $P = \langle x, y, z \mid yz \xrightarrow{\alpha} -x^2, zy \xrightarrow{\beta} -\mu x^2 \rangle$ presenting the algebra $A = \mathbb{K}\langle x, y, z \mid x^2 + yz = 0, x^2 + \lambda zy = 0 \rangle$, with $\mu := \lambda^{-1}$.

- It terminates, using the deglex order generated by $z > y > x$.

- Squier's completion:



- $\langle x, y, z \mid \alpha, \beta, \gamma, \delta \mid A, B, C, D \rangle$ is a coherent presentation of A .

- **Example:** The linear 2-polygraph $P = \langle x, y, z \mid xyz \Rightarrow x^3 + y^3 + z^3 \rangle$ is convergent and admits an empty homotopy basis.