Linear polygraphs.

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From Gröbner bases to linear polygraphs

- Last week was presented the approach of linear rewriting using the theory of **Gröbner bases**.
  - Orientation of relations depend on an ambient monomial order, that is a well-founded total order such that $s(f) > t(f)$ for any rule f and $uvw > uv'w$ for any monomials u, v, w such that $v > v'$.

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- Introduction of **linear polygraphs**:

CONVERGENT PRESENTATIONS AND POLYGRAPHIC RESOLUTIONS OF ASSOCIATIVE ALGEBRAS

YVES GUIRAUD ERIC HOFFBECK PHILIPPE MALBOS

Abstract – Several constructive homological methods based on noncommutative Gröbner bases are known to compute free resolutions of associative algebras. In particular, these methods relate the Koszul property for an associative algebra to the existence of a quadratic Gröbner basis of its ideal of relations. In this article, using a higher-dimensional rewriting theory approach, we give several improvements of these methods. We define polygraphs for associative algebras as higher-dimensional linear rewriting systems that generalise the notion of noncommutative Gröbner bases, and allow more possibilities of termination orders than those associated to monomial orders. We introduce polygraphic resolutions of associative algebras, giving a categorical description of higher-dimensional syzygies for presentations of algebras. We show how to compute polygraphic resolutions starting from a convergent presentation, and how these resolutions can be linked with the Koszul property.

Keywords – Higher-dimensional associative algebras, confluence and termination, linear rewriting, polygraphs, free resolutions, Koszulness.

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A toy example

- Consider an associative algebra $A = \mathbb{K}\langle x,\ y,\ z \mid xyz - x^3 - y^3 - z^3\rangle$, *i.e.* A is the algebra generated by x,y and z quotiented by the ideal generated by $xyz - x^3 - y^3 - z^3$.

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- If we orient the relation as $xyz \Rightarrow x^3 + y^3 + z^3$, this **can not** be compatible with a monomial order.
  - Suppose such an order $\prec$ exists.
  - Since $\prec$ is total, one of x,y,z is greater than the other two. Suppose it is x.
  - Then $x \succ y$ implies $x^2 \succ yx$ and $x \succ z$ implies $yx \succ yz$.
  - Hence $x^2 \succ yz$, and $x^3 \succ xyz$.
  - We show that any x^3, y^3 and z^3 is greater than xyz.

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  - Consider the map $\Phi : \{x, y, z\}^* \to \mathbb{N}$ defined by

$$\Phi(u) := 3 \times \text{ number of } xyz \text{ in } u + \text{number of } y \text{ in } u.$$

  - $\Phi(uxyzv) > \Phi(ux^3v)$, $\Phi(uxyzv) > \Phi(uy^3v)$, $\Phi(uxyzv) > \Phi(uz^3v)$ for any $u, v \in \{x, y, z\}^*$.

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  - Therefore, if we have a rewriting step $f \Rightarrow \sum_i \lambda_i f_i$, we have $\Phi(f) > \Phi(f_i)$.
  - There cannot exist an infinite rewriting sequence $f \Rightarrow f' \Rightarrow f'' \Rightarrow \dots$ in P, otherwise there would be a strictly decreasing infinite sequence of natural numbers

$$\Phi(f) > \Phi(f') > \Phi(f'') > \dots$$

Outline

I. Linear 2-polygraphs

II. The linear critical branching theorem

III. Squier's coherence theorem

IV. Higher-dimensional linear polyraphs

I. Linear 2-polygraphs

Algebroids

- There are two ways to see an associative algebra over a field $\mathbb{K}$:
  1) As a monoid object in the category $\mathbf{Vect}_{\mathbb{K}}$ (that is monoidal with product given by $\otimes$).
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$$\text{Monoids} \leftrightsquigarrow 1\text{-categories with only one } 0\text{-cell.}$$

$$\text{Associative algebras} \leftrightsquigarrow 1\text{-}\textbf{algebroids} \text{ with only one } 0\text{-cell.}$$

- A 1-algebroid over a field $\mathbb{K}$ is a 1-category enriched over the category $\mathbf{Vect}_{\mathbb{K}}$ of $\mathbb{K}$-vector spaces.
- Explicitely, it is given by:
  - a set of 0-cells A_0,
  - for every 0-cells p and q, a $\mathbb{K}$-vector space $A(p,q)$, whose elements are the 1-cells of A.
  - for any 0-cells p,q and r, there is a $\mathbb{K}$-linear map $\star_0 : A(p,q) \otimes A(q,r) \to A(p,r)$, and we denote $\star_0(f \otimes g)$ by fg.
  - this composition is associtative: $(fg)h = f(gh)$, and unitary: $1_p f = f = f 1_q$ for any $f \in A(p,q)$.

Linear 2-polygraphs

- Let (P_0, P_1) be a 1-polygraph, *i.e.*, a directed graph with source and target maps s_0, t_0.
- The **free 1-algebroid** over P is the 1-algebroid P_1^ℓ defined by:
  - $(P_1)_0^\ell = P_0$,
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- A **linear 2-polygraph** is a triple (P_0, P_1, P_2) made of
  - a 1-polygraph (P_0, P_1),
  - a cellular extension P_2 of the free 1-algebroid P_1^ℓ, with source and target maps s_1,t_1 satisfying globular relations:

$$s_0 s_1 = s_0 t_1, \quad y_0 t_1 = t_0 t_1.$$

- An element of P_2 is called a **rewriting rule**, and is depicted by

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- **From now on, we consider linear 2-polygraphs with only one 0-cell.**
- The **ideal** of a linear 2-polygraph P is the two-sided ideal of the algebra P_1^ℓ generated by

$$\{s_1(\alpha) - t_1(\alpha) \mid \alpha \in P_2\}$$

  The **algebra presented by** P is the $\mathbb{K}$-algebra given by $P_1^\ell \,/\, I(P)$.

2-Algebras

- A **2-algebra** is an internal category in the category $\mathbf{Alg}_{\mathbb{K}}$ of associative $\mathbb{K}$-algebras.

- Explicitely, it is given by a diagram in **Alg**:

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- The product of two 2-cells a and b in A_2 is denoted by ab.

- The linear structure and product in the algebra $A_2 \times_{A_1} A_2$ are given by:

$$(a, a') + (b, b') = (a + b, a' + b'),\ \lambda(a, a') = (\lambda a, \lambda a'),\ (a, a')(b, b') = (ab, a'b')$$

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- Elements of A_1 are called **1-cells of A**, and are pictured as:

$$\begin{tikzcd} \bullet \arrow[r, bend left, "f"] \arrow[r, "g"] \arrow[r, bend right, "h"'] & \bullet \end{tikzcd}$$

- Elements of A_2 are called **2-cells of A**, and are pictured as

$$\begin{tikzcd} \bullet \arrow[r, bend left, "s_1(\alpha)", ""{name=U, below}] \arrow[r, bend right, "t_1(\alpha)"', ""{name=D, above}] \arrow[Rightarrow, from=U, to=D, "\alpha"] & \bullet \end{tikzcd}$$

Properties of sources, targets and compositions of a 2-algebra

- For any 2-cells a, b and any $\lambda, \mu \in \mathbb{K}$, we have

$$\partial_1(ab) = \partial_1(a)\partial_1(b), \quad \partial_1(\lambda a + \mu b) = \lambda\partial_1(a) + \mu\partial_1(b) \text{ for } \partial \in \{s, t\}.$$

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- **Properties of $\star_1$-composition:** Given two 2-cells as follows:

$$\begin{tikzcd}
\bullet \arrow[r, bend left=70, "f"] \arrow[r, "f'" description] \arrow[r, bend right=70, "f''"'] \arrow[r, phantom, shift left=4, "\Downarrow a"] \arrow[r, phantom, shift right=4, "\Downarrow a'"] & \bullet & \bullet \arrow[r, bend left=70, "g"] \arrow[r, "g'" description] \arrow[r, bend right=70, "g''"'] \arrow[r, phantom, shift left=4, "\Downarrow b"] \arrow[r, phantom, shift right=4, "\Downarrow b'"] & \bullet & \rightsquigarrow & \bullet \arrow[r, bend left=70, "f+g"] \arrow[r, "f'+g'" description] \arrow[r, bend right=70, "f''+g''"'] \arrow[r, phantom, shift left=4, "\Downarrow a+b"] \arrow[r, phantom, shift right=4, "\Downarrow a'+b'"] & \bullet
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$$(a+b) \star_1 (a' + b') = a \star_1 a' + b \star_1 b'$$

$$(\lambda a) \star_1 (\lambda a') = \lambda(a \star_1 a') \qquad (\lambda a \star_1 \mu a' \text{ is not defined if } \lambda \neq \mu)$$

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- **Properties of $\star_1$-composition:** Given two 2-cells as follows:

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\bullet \arrow[r, bend left=60, "f"] \arrow[r, bend left=30, phantom, "\Downarrow a"] \arrow[r, "f'" description] \arrow[r, bend right=30, phantom, "\Downarrow a'"] \arrow[r, bend right=60, "f''"'] & \bullet & \bullet \arrow[r, bend left=60, "g"] \arrow[r, bend left=30, phantom, "\Downarrow b"] \arrow[r, "g'" description] \arrow[r, bend right=30, phantom, "\Downarrow b'"] \arrow[r, bend right=60, "g''"'] & \bullet & \rightsquigarrow & \bullet \arrow[r, bend left=60, "f+g"] \arrow[r, bend left=30, phantom, "\Downarrow a+b"] \arrow[r, "f'+g'" description] \arrow[r, bend right=30, phantom, "\Downarrow a'+b'"] \arrow[r, bend right=60, "f''+g''"'] & \bullet
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- **Some identities:**
  - For any 1-composable a and a', we have $a \star_1 a' = a + a' - t_1(a)$,
  - For any 2-cell a, we have $a^- = -a + s_1(a) + t_1(a)$, so that any 2-cell is invertible for the $\star_1$-composition.
  - For any 2-cell a and b in A, we have

$$ab = as_1(b) + t_1(a)b - t_1(a)s_1(b) \underset{\mathrm{Eq}_{a,b}}{=} s_1(a)b + at_1(b) - s_1(a)t_1(b)$$

The free 2-algebra on a linear 2-polygraph

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  $$\{Eq_{a,b} \mid a, b \in P_1^\ell \otimes \mathbb{K}P_2 \otimes P_1^\ell\}.$$
  - It has a structure of algebra, with product defined by
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$$\begin{tikzcd} \bullet \arrow[r, "f"] & \bullet \arrow[r, bend left, "\lambda s_1(\alpha)"] \arrow[r, bend right, "\lambda t_1(\alpha)"'] \arrow[r, phantom, "\Downarrow \lambda\alpha"] & \bullet \arrow[r, "g"] & \bullet \end{tikzcd} + \begin{tikzcd} \bullet \arrow[r, "h"] & \bullet \end{tikzcd}$$

The free 2-algebra on a linear 2-polygraph

- Given a linear 2-polygraph P, the **free 2-algebra over P** is the 2-algebra given by the following diagram

$$\begin{tikzcd} P_1^\ell \arrow[r, bend left, "i_2"] & P_2^\ell \arrow[l, shift right, "s_1"'] \arrow[l, shift left, "t_1"] & P_2^\ell \times_{P_1^\ell} P_2^\ell \arrow[l, "\star_1"] \end{tikzcd}$$

where:

  - P_2^ℓ is defined as the quotient of the P_1^ℓ-bimodule $(P_1^\ell \otimes \mathbb{K}P_2 \otimes P_1^\ell) \oplus P_1^\ell$ quotiented by the equivalence relation generated by
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- The sphere correspond to the "monomial" place where we will apply rewriting steps inside a polynomial.

Monomials and rewriting steps

- A **monomial** in P_2^ℓ is a 1-cell in the free monoid P_1^* over P_1.
  - The monomials of P_2^ℓ form a linear basis of the algebra P_1^ℓ.
  - Every 1-cell $f \neq 0$ of P_1^ℓ can be uniquely written as $f = \lambda_1 u_1 + \cdots + \lambda_p u_p$ with $\lambda_i \in \mathbb{K}\backslash\{0\}$ and u_i monomials.

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- A 2-cell of P_2^ℓ with that shape but without the green condition is called **elementary**.

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$$P_1^\ell = P_1^{\mathrm{nf}} + I(P), \qquad f = \widehat{f} + (f - \widehat{f}).$$

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- The decomposition is not direct in general: consider $P = \langle x, y \mid x^2 \overset{\beta}{\Rightarrow} xy \rangle$.

$$\begin{tikzcd} & xyx \\ x^3 \arrow[ur, Rightarrow, bend left, "\beta x"] \arrow[dr, Rightarrow, bend right, "x\beta"'] & \\ & x^2y \arrow[r, Rightarrow, "\beta y"'] & xy^2 \end{tikzcd}$$

$$xyx - xy^2 = -(x^2 - xy)x + x(x^2 - xy) + (x^2 - xy)y$$

Confluence and linear bases

- **Theorem:** The following conditions are equivalent:
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with g', h' reduced. Then, $g' - h'$ is reduced.

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- **Theorem:** Let A be an algebra and P be a convergent linear 2-polygraph presenting A. The set P_1^{mnf} of monomials of P_1^ℓ in normal form w.r.t P is a **linear basis** of A.

II. The linear critical branching theorem

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$$\lambda \begin{tikzcd} \bullet \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & \bullet \end{tikzcd} + \begin{tikzcd} \bullet \arrow[r, "h"] & \bullet \end{tikzcd}$$

  where α is a 2-monomial, $\lambda \in \mathbb{K}$, g is a 1-cell of P_1^ℓ such that $\mathbf{f} \notin \mathrm{Supp}(\mathbf{h})$.

- If P is convergent, we have the direct sum decomposition

$$P_1^\ell = P_1^{\mathrm{nf}} \oplus I(P)$$

  and thus the monomials in normal form give a linear basis of the algebra $\overline{P} := P_1^\ell \;/\; I(P)$.

- Associative algebras over a field $\mathbb{K}$ are presented by **linear 2-polygraps**. These are triples $P = (P_0, P_1, P_2)$ where:
  - $P_0 = \{\bullet\}$,
  - generating 1-cells in $P_1 \ni x, y, z, \ldots$ $\rightsquigarrow P_1^\ell \ni xyz - x^3 - y^3 - z^3$,
  - generating 2-cells (or rewriting rules) in $P_2 \ni xyz \Rightarrow x^3 + y^3 + z^3$

$$\rightsquigarrow P_2^\ell \ni x^3yzx + xz + z^4 \Rightarrow x^6 + x^2y^3x + x^2z^3x + xz + z^4.$$

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- If P is a linear 2-polygraph, $(P_1^\ell, \Rightarrow_{\mathrm{stp}})$ gives an abstract rewriting system.
  - As opposed to set-theoretical context, we do not consider all the 2-cells of P_2^ℓ.
  - **Newman lemma:** If P is terminating, confluence and local confluence are equivalent properties.

Local branchings

- Local branchings of a left-monomial linear 2-polygraph are split into the following four families:

1) **Asphericial**: $\begin{tikzcd} \lambda f + h \arrow[rr, bend left, "\lambda a + h"] \arrow[rr, bend right, "\lambda a + h"'] & & \lambda g + h \end{tikzcd}$

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Local confluence from critical confluence

- **Example:** Consider the linear 2-polygraph $P = \langle x, y, z, t \mid xy \overset{\alpha}{\Rightarrow} xz,\ zt \overset{\beta}{\Rightarrow} 2yt \rangle$.
  - It has no critical branching, but it has a non-confluent additive branching:

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$$\begin{tikzcd} & 2xzt \\ xyt + xzt \arrow[ur, Rightarrow, bend left, "\alpha t + xzt"] \arrow[dr, Rightarrow, bend right, "xyt + x\beta"'] & \\ & 3xyt \end{tikzcd}$$

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xyt + xzt \arrow[ur, Rightarrow, bend left, "\alpha t + xzt"] \arrow[dr, Rightarrow, bend right, "xyt + x\beta"'] & & xzt + 2xyt \\
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xyt + xzt \arrow[ur, Rightarrow, bend left, "\alpha t + xzt"] \arrow[dr, Rightarrow, bend right, "xyt + x\beta"'] & & xzt + 2xyt & & \\
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 & 2xzt \arrow[ur, Rightarrow, "2x\beta"] \arrow[dr, Rightarrow, dotted, "xzt + x\beta"] & & & \\
xyt + xzt \arrow[ur, Rightarrow, "\alpha t + xzt"] \arrow[dr, Rightarrow, "xyt + x\beta"'] & & xzt + 2xyt & & \\
 & 3xyt \arrow[ur, Rightarrow, dotted, "\alpha t + 2xyt"'] \arrow[dr, Rightarrow, "3\alpha t"'] & & & \\
 & & 3xzt \arrow[r, Rightarrow, "3x\beta"'] & 6xyt \arrow[r, Rightarrow, "6\alpha t"'] & \cdots
\end{tikzcd}
$$

Local confluence from critical confluence

- **Example:** Consider the linear 2-polygraph $P = \langle x, y, z, t \mid xy \overset{\alpha}{\Rightarrow} xz,\ zt \overset{\beta}{\Rightarrow} 2yt \rangle$.
  - It has no critical branching, but it has a non-confluent additive branching:

$$\begin{tikzcd}
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xyt + xzt \arrow[ur, Rightarrow, bend left, "\alpha t + xzt"] \arrow[dr, Rightarrow, bend right, "xyt + x\beta"'] & & xzt + 2xyt & & \\
 & 3xyt \arrow[ur, Rightarrow, dotted, "\alpha t + 2xyt"'] \arrow[dr, Rightarrow, bend right, "3\alpha t"'] & & & \\
 & & 3xzt \arrow[r, Rightarrow, "3x\beta"'] & 6xyt \arrow[r, Rightarrow, "6\alpha t"'] & \cdots
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  - We need termination to ensure confluence of additive branchings.

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- **Example:** Consider the linear 2-polygraph $\langle x, y, z \mid xy \overset{\alpha}{\Rightarrow} 2x,\ yz \overset{\beta}{\Rightarrow} z \rangle$.
  - It terminates, but has a nonconfluent Peiffer branching:

$$
\begin{tikzcd}
 & 6xz & 3xz \\
 & 3xyz \arrow[u, Rightarrow, "3\alpha z"] \arrow[ur, Rightarrow, "3x\beta"] \arrow[dr, Rightarrow, dotted, "2x\beta + xyz"] & \\
xyyz + xyz \arrow[ur, Rightarrow, bend left, "\alpha yz + xyz"] \arrow[dr, Rightarrow, bend right, "xy\beta + xyz"'] & & 2xz + xyz \\
 & 2xyz \arrow[ur, Rightarrow, dotted, "\alpha z + xyz"'] \arrow[d, Rightarrow, "2x\beta"'] \arrow[dr, Rightarrow, "2\alpha z"'] & \\
 & 4xz & 2xz
\end{tikzcd}
$$

Local confluence from critical confluence

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 & 2xyz \arrow[d, Rightarrow, "2x\beta"'] \arrow[ur, Rightarrow, dotted, "\alpha z + xyz"'] \arrow[dr, Rightarrow, "2\alpha z"'] & \\
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\end{tikzcd}
$$

  - Critical confluence is needed to ensure confluence of Peiffer branchings.

The linear critical branching theorem

- **Theorem:** A **terminating** left-monomial linear 2-polygraph is locally confluent if and only if all its critical branchings are confluent.

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$$\begin{tikzcd} f \arrow[rr, Rightarrow, bend left, "\alpha"] \arrow[dr, Rightarrow, "\beta"'] & & g \arrow[dl, Rightarrow, "\gamma"] \\ & h & \end{tikzcd}$$

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- **Example:** Let P be a linear 2-polygraph and $\alpha : f \Rightarrow g$ be a 2-cell.

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$$\begin{tikzcd} & & f_p \\ & f_1 \arrow[ur, Rightarrow, bend left, "\alpha_2 \star_0 \cdots \star_0 \alpha_p"] \arrow[dr, Rightarrow, "\gamma_1"] & \\ f_0 \arrow[ur, Rightarrow, bend left, "\alpha_1"] \arrow[rr, Rightarrow, bend right, "\beta_1"'] & & g_1 \end{tikzcd}$$

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$$\begin{tikzcd} & & f_p \arrow[dr, Rightarrow, bend left, "\gamma_2"] & \\ & f_1 \arrow[ur, Rightarrow, bend left, "\alpha_2 \star_0 \cdots \star_0 \alpha_p"] \arrow[rr, Rightarrow, "\beta_2"] \arrow[dr, Rightarrow, "\gamma_1"] & & g_2 \\ f_0 \arrow[ur, Rightarrow, bend left, "\alpha_1"] \arrow[rr, Rightarrow, bend right, "\beta_1"'] & & g_1 & \end{tikzcd}$$

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- **Step 1:** Additive branchings are confluent.

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$$
\begin{tikzcd}
 & \lambda f' + \mu g + h \\
\lambda f + \mu g + h \arrow[ur, Rightarrow, bend left, "\lambda a + \mu g + h"] \arrow[dr, Rightarrow, bend right, "\lambda f + \mu b + h"'] & \\
 & \lambda f + \mu g' + h
\end{tikzcd}
$$

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$$
\begin{tikzcd}
& \lambda f' + \mu g + h \arrow[dr, Rightarrow, dotted, "\lambda f' + \mu b + h"] & \\
\lambda f + \mu g + h \arrow[ur, Rightarrow, bend left, "\lambda a + \mu g + h"] \arrow[dr, Rightarrow, bend right, "\lambda f + \mu b + h"'] & & \lambda f' + \mu g' + h \\
& \lambda f + \mu g' + h \arrow[ur, Rightarrow, dotted, "\lambda a + \mu g' + h"'] &
\end{tikzcd}
$$

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$$
\begin{tikzcd}
 & \lambda f' + \mu g + h \arrow[rr, Rightarrow, "a'_1"] \arrow[dr, Rightarrow, dotted, "\lambda f' + \mu b + h"] & & k \\
\lambda f + \mu g + h \arrow[ur, Rightarrow, bend left, "\lambda a + \mu g + h"] \arrow[dr, Rightarrow, bend right, "\lambda f + \mu b + h"'] & & \lambda f' + \mu g' + h \arrow[ur, Rightarrow, "m"] \arrow[dr, Rightarrow, "n"] & \\
 & \lambda f + \mu g' + h \arrow[rr, Rightarrow, "b'_1"] \arrow[ur, Rightarrow, dotted, "\lambda a + \mu g' + h"] & & k'
\end{tikzcd}
$$

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$$
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\lambda f + \mu g + h \arrow[ur, Rightarrow, "\lambda a + \mu g + h"] \arrow[dr, Rightarrow, "\lambda f + \mu b + h"'] & & \lambda f' + \mu g' + h \arrow[ur, Rightarrow, "m"] \arrow[dr, Rightarrow, "n"'] & \text{Induction.} & p \\
 & \lambda f + \mu g' + h \arrow[rr, Rightarrow, "b'_1"'] \arrow[ur, dotted, Rightarrow, "\lambda a + \mu g' + h"'] & & k' \arrow[ur, Rightarrow, "b'_2"'] &
\end{tikzcd}
$$

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$$
\begin{tikzcd}
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\lambda f + \mu g + h \arrow[ur, Rightarrow, bend left, "\lambda a + \mu g + h"] \arrow[dr, Rightarrow, bend right, "\lambda f + \mu b + h"'] & & \lambda f' + \mu g' + h \arrow[ur, Rightarrow, "m"] \arrow[dr, Rightarrow, "n"'] & \text{Induction.} & {} \\
 & \lambda f + \mu g' + h \arrow[rr, Rightarrow, "b'_1"'] \arrow[ur, Rightarrow, dotted, "\lambda a + \mu g' + h"'] & & k' \arrow[ur, Rightarrow, bend right, "b'_2"'] &
\end{tikzcd}
$$

- **Step 2:** Peiffer are confluent.

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$$
\begin{tikzcd}
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\lambda f + \mu g + h \arrow[ur, Rightarrow, bend left, "\lambda a + \mu g + h"] \arrow[dr, Rightarrow, bend right, "\lambda f + \mu b + h"'] & & \lambda f' + \mu g' + h \arrow[ur, Rightarrow, "m"] \arrow[dr, Rightarrow, "n"'] & \text{Induction.} & {} \\
& \lambda f + \mu g' + h \arrow[rr, Rightarrow, "b'_1"'] \arrow[ur, dotted, Rightarrow, "\lambda a + \mu g' + h"'] & & k' \arrow[ur, Rightarrow, bend right, "b'_2"'] &
\end{tikzcd}
$$

- **Step 2:** Peiffer are confluent.

$$
\begin{tikzcd}
& \lambda f' g + h \\
\lambda fg + h \arrow[ur, Rightarrow, bend left, "\lambda ag + h"] \arrow[dr, Rightarrow, bend right, "\lambda fb + h"'] & \\
& \lambda fg' + h
\end{tikzcd}
$$

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$$
\begin{tikzcd}
& \lambda f' + \mu g + h \arrow[rr, Rightarrow, "a'_1"] \arrow[dr, Rightarrow, dotted, "\lambda f' + \mu b + h"] & & k \arrow[dr, Rightarrow, bend left, "a'_2"] & \\
\lambda f + \mu g + h \arrow[ur, Rightarrow, bend left, "\lambda a + \mu g + h"] \arrow[dr, Rightarrow, bend right, "\lambda f + \mu b + h"'] & & \lambda f' + \mu g' + h \arrow[ur, Rightarrow, "m"] \arrow[dr, Rightarrow, "n"'] \arrow[rr, phantom, "\text{Induction.}"] & & {} \\
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$$
\begin{tikzcd}
& \lambda f' g + h \arrow[dr, Rightarrow, dotted, "\lambda f' b + h"] & \\
\lambda f g + h \arrow[ur, Rightarrow, bend left, "\lambda a g + h"] \arrow[dr, Rightarrow, bend right, "\lambda f b + h"'] & & \lambda f' g' + h \\
& \lambda f g' + h \arrow[ur, Rightarrow, dotted, "\lambda a g' + h"'] &
\end{tikzcd}
$$

The linear critical branching theorem

- **Step 1:** Additive branchings are confluent.

$$
\begin{tikzcd}
 & \lambda f' + \mu g + h \arrow[rr, Rightarrow, "a'_1"] \arrow[dr, dotted, Rightarrow, "\lambda f' + \mu b + h"] & & k \arrow[dr, Rightarrow, bend left, "a'_2"] & \\
\lambda f + \mu g + h \arrow[ur, Rightarrow, bend left, "\lambda a + \mu g + h"] \arrow[dr, Rightarrow, bend right, "\lambda f + \mu b + h"'] & & \lambda f' + \mu g' + h \arrow[ur, Rightarrow, "m"] \arrow[dr, Rightarrow, "n"'] & \text{Induction.} & \\
 & \lambda f + \mu g' + h \arrow[rr, Rightarrow, "b'_1"'] \arrow[ur, dotted, Rightarrow, "\lambda a + \mu g' + h"'] & & k' \arrow[ur, Rightarrow, bend right, "b'_2"'] &
\end{tikzcd}
$$

- **Step 2:** Peiffer are confluent.

$$
\begin{tikzcd}
 & \lambda f' g + h \arrow[rr, Rightarrow, "a'_1"] \arrow[dr, dotted, Rightarrow, "\lambda f' b + h"] & & k \\
\lambda f g + h \arrow[ur, Rightarrow, bend left, "\lambda a g + h"] \arrow[dr, Rightarrow, bend right, "\lambda f b + h"'] & & \lambda f' g' + h \arrow[ur, Rightarrow, "m"] \arrow[dr, Rightarrow, "n"'] & \\
 & \lambda f g' + h \arrow[rr, Rightarrow, "b'_1"'] \arrow[ur, dotted, Rightarrow, "\lambda a g' + h"'] & & k'
\end{tikzcd}
$$

The linear critical branching theorem

- **Step 1:** Additive branchings are confluent.

$$
\begin{tikzcd}
& \lambda f' + \mu g + h \arrow[rr, Rightarrow, "a'_1"] \arrow[dr, Rightarrow, dotted, "\lambda f' + \mu b + h"] & & k \arrow[dr, Rightarrow, bend left, "a'_2"] & \\
\lambda f + \mu g + h \arrow[ur, Rightarrow, bend left, "\lambda a + \mu g + h"] \arrow[dr, Rightarrow, bend right, "\lambda f + \mu b + h"'] & & \lambda f' + \mu g' + h \arrow[ur, Rightarrow, "m"] \arrow[dr, Rightarrow, "n"'] & \text{Induction.} & {} \\
& \lambda f + \mu g' + h \arrow[rr, Rightarrow, "b'_1"'] \arrow[ur, Rightarrow, dotted, "\lambda a + \mu g' + h"'] & & k' \arrow[ur, Rightarrow, bend right, "b'_2"'] &
\end{tikzcd}
$$

- **Step 2:** Peiffer are confluent.

$$
\begin{tikzcd}
& \lambda f' g + h \arrow[rr, Rightarrow, "a'_1"] \arrow[dr, Rightarrow, dotted, "\lambda f' b + h"] & & k \arrow[dr, Rightarrow, bend left, "a'_2"] & \\
\lambda fg + h \arrow[ur, Rightarrow, bend left, "\lambda ag + h"] \arrow[dr, Rightarrow, bend right, "\lambda fb + h"'] & & \lambda f' g' + h \arrow[ur, Rightarrow, "m"] \arrow[dr, Rightarrow, "n"'] & \text{Induction.} & p \\
& \lambda fg' + h \arrow[rr, Rightarrow, "b'_1"'] \arrow[ur, Rightarrow, dotted, "\lambda ag' + h"'] & & k' \arrow[ur, Rightarrow, bend right, "b'_2"'] &
\end{tikzcd}
$$

The linear critical branching theorem

- **Step 1:** Additive branchings are confluent.

$$
\begin{tikzcd}
 & \lambda f' + \mu g + h \arrow[rr, Rightarrow, "a'_1"] \arrow[dr, Rightarrow, dotted, "\lambda f' + \mu b + h"] & & k \arrow[dr, Rightarrow, bend left, "a'_2"] & \\
\lambda f + \mu g + h \arrow[ur, Rightarrow, bend left, "\lambda a + \mu g + h"] \arrow[dr, Rightarrow, bend right, "\lambda f + \mu b + h"'] & & \lambda f' + \mu g' + h \arrow[ur, Rightarrow, "m"] \arrow[dr, Rightarrow, "n"'] & \text{Induction.} & \\
 & \lambda f + \mu g' + h \arrow[rr, Rightarrow, "b'_1"'] \arrow[ur, Rightarrow, dotted, "\lambda a + \mu g' + h"] & & k' \arrow[ur, Rightarrow, bend right, "b'_2"'] &
\end{tikzcd}
$$

- **Step 2:** Peiffer are confluent.

$$
\begin{tikzcd}
 & \lambda f' g + h \arrow[rr, Rightarrow, "a'_1"] \arrow[dr, Rightarrow, dotted, "\lambda f' b + h"] & & k \arrow[dr, Rightarrow, bend left, "a'_2"] & \\
\lambda f g + h \arrow[ur, Rightarrow, bend left, "\lambda a g + h"] \arrow[dr, Rightarrow, bend right, "\lambda f b + h"'] & & \lambda f' g' + h \arrow[ur, Rightarrow, "m"] \arrow[dr, Rightarrow, "n"'] & \text{Induction.} & p \\
 & \lambda f g' + h \arrow[rr, Rightarrow, "b'_1"'] \arrow[ur, Rightarrow, dotted, "\lambda a g' + h"] & & k' \arrow[ur, Rightarrow, bend right, "b'_2"'] &
\end{tikzcd}
$$

- **Step 3:** Overlappings are confluent.

$$
\begin{tikzcd}
 & \lambda a + c \\
\lambda u + c \arrow[ur, Rightarrow, bend left, "\lambda f + c"] \arrow[dr, Rightarrow, bend right, "\lambda g + c"'] & \\
 & \lambda b + c
\end{tikzcd}
$$

The linear critical branching theorem

- **Step 1:** Additive branchings are confluent.

$$\begin{tikzcd}
 & \lambda f' + \mu g + h \arrow[rr, Rightarrow, "a'_1"] \arrow[dr, dotted, Rightarrow, "\lambda f' + \mu b + h"] & & k \arrow[dr, Rightarrow, bend left, "a'_2"] & \\
\lambda f + \mu g + h \arrow[ur, Rightarrow, bend left, "\lambda a + \mu g + h"] \arrow[dr, Rightarrow, bend right, "\lambda f + \mu b + h"'] & & \lambda f' + \mu g' + h \arrow[ur, Rightarrow, "m"] \arrow[dr, Rightarrow, "n"'] \arrow[r, phantom, "\text{Induction.}"] & & {} \\
 & \lambda f + \mu g' + h \arrow[rr, Rightarrow, "b'_1"'] \arrow[ur, dotted, Rightarrow, "\lambda a + \mu g' + h"'] & & k' \arrow[ur, Rightarrow, bend right, "b'_2"'] &
\end{tikzcd}$$

- **Step 2:** Peiffer are confluent.

$$\begin{tikzcd}
 & \lambda f' g + h \arrow[rr, Rightarrow, "a'_1"] \arrow[dr, dotted, Rightarrow, "\lambda f' b + h"] & & k \arrow[dr, Rightarrow, bend left, "a'_2"] & \\
\lambda f g + h \arrow[ur, Rightarrow, bend left, "\lambda a g + h"] \arrow[dr, Rightarrow, bend right, "\lambda f b + h"'] & & \lambda f' g' + h \arrow[ur, Rightarrow, "m"] \arrow[dr, Rightarrow, "n"'] \arrow[r, phantom, "\text{Induction.}"] & & p \\
 & \lambda f g' + h \arrow[rr, Rightarrow, "b'_1"'] \arrow[ur, dotted, Rightarrow, "\lambda a g' + h"'] & & k' \arrow[ur, Rightarrow, bend right, "b'_2"'] &
\end{tikzcd}$$

- **Step 3:** Overlappings are confluent.

$$\begin{tikzcd}
 & \lambda a + c \\
\lambda u + c \arrow[ur, Rightarrow, bend left, "\lambda k f_0 k' + c"] \arrow[dr, Rightarrow, bend right, "\lambda k g_0 g' + c"'] & \\
 & \lambda b + c
\end{tikzcd}$$

The linear critical branching theorem

- **Step 1:** Additive branchings are confluent.

$$
\begin{tikzcd}
 & \lambda f' + \mu g + h \arrow[rr, Rightarrow, "a'_1"] \arrow[dr, dotted, Rightarrow, "\lambda f' + \mu b + h"] & & k \arrow[dr, Rightarrow, bend left, "a'_2"] & \\
\lambda f + \mu g + h \arrow[ur, Rightarrow, bend left, "\lambda a + \mu g + h"] \arrow[dr, Rightarrow, bend right, "\lambda f + \mu b + h"'] & & \lambda f' + \mu g' + h \arrow[ur, Rightarrow, "m"] \arrow[dr, Rightarrow, "n"'] & \text{Induction.} & {} \\
 & \lambda f + \mu g' + h \arrow[rr, Rightarrow, "b'_1"'] \arrow[ur, dotted, Rightarrow, "\lambda a + \mu g' + h"] & & k' \arrow[ur, Rightarrow, bend right, "b'_2"'] &
\end{tikzcd}
$$

- **Step 2:** Peiffer are confluent.

$$
\begin{tikzcd}
 & \lambda f' g + h \arrow[rr, Rightarrow, "a'_1"] \arrow[dr, dotted, Rightarrow, "\lambda f' b + h"] & & k \arrow[dr, Rightarrow, bend left, "a'_2"] & \\
\lambda f g + h \arrow[ur, Rightarrow, bend left, "\lambda a g + h"] \arrow[dr, Rightarrow, bend right, "\lambda f b + h"'] & & \lambda f' g' + h \arrow[ur, Rightarrow, "m"] \arrow[dr, Rightarrow, "n"'] & \text{Induction.} & p \\
 & \lambda f g' + h \arrow[rr, Rightarrow, "b'_1"'] \arrow[ur, dotted, Rightarrow, "\lambda a g' + h"] & & k' \arrow[ur, Rightarrow, bend right, "b'_2"'] &
\end{tikzcd}
$$

- **Step 3:** Overlappings are confluent.

$$
\begin{tikzcd}
 & \lambda a + c \arrow[dr, dotted, Rightarrow, "\lambda k f'_0 k' + c"] & \\
\lambda u + c \arrow[ur, Rightarrow, bend left, "\lambda k f_0 k' + c"] \arrow[dr, Rightarrow, bend right, "\lambda k g_0 g' + c"'] & & \lambda e + c \\
 & \lambda b + c \arrow[ur, dotted, Rightarrow, "\lambda k g'_0 k' + c"'] &
\end{tikzcd}
$$

The linear critical branching theorem

- **Step 1:** Additive branchings are confluent.

$$
\begin{tikzcd}
 & \lambda f' + \mu g + h \arrow[rr, Rightarrow, "a'_1"] \arrow[dr, dotted, "\lambda f' + \mu b + h"] & & k \arrow[dr, Rightarrow, bend left, "a'_2"] & \\
\lambda f + \mu g + h \arrow[ur, Rightarrow, bend left, "\lambda a + \mu g + h"] \arrow[dr, Rightarrow, bend right, "\lambda f + \mu b + h"'] & & \lambda f' + \mu g' + h \arrow[ur, Rightarrow, "m"] \arrow[dr, Rightarrow, "n"'] & \text{Induction.} & {} \\
 & \lambda f + \mu g' + h \arrow[rr, Rightarrow, "b'_1"'] \arrow[ur, dotted, "\lambda a + \mu g' + h"] & & k' \arrow[ur, Rightarrow, bend right, "b'_2"'] &
\end{tikzcd}
$$

- **Step 2:** Peiffer are confluent.

$$
\begin{tikzcd}
 & \lambda f' g + h \arrow[rr, Rightarrow, "a'_1"] \arrow[dr, dotted, "\lambda f' b + h"] & & k \arrow[dr, Rightarrow, bend left, "a'_2"] & \\
\lambda f g + h \arrow[ur, Rightarrow, bend left, "\lambda a g + h"] \arrow[dr, Rightarrow, bend right, "\lambda f b + h"'] & & \lambda f' g' + h \arrow[ur, Rightarrow, "m"] \arrow[dr, Rightarrow, "n"'] & \text{Induction.} & p \\
 & \lambda f g' + h \arrow[rr, Rightarrow, "b'_1"'] \arrow[ur, dotted, "\lambda a g' + h"] & & k' \arrow[ur, Rightarrow, bend right, "b'_2"'] &
\end{tikzcd}
$$

- **Step 3:** Overlappings are confluent.

$$
\begin{tikzcd}
 & \lambda a + c \arrow[rr, Rightarrow, "a'_1"] \arrow[dr, dotted, "\lambda k f'_0 k' + c"] & & l \\
\lambda u + c \arrow[ur, Rightarrow, bend left, "\lambda k f_0 k' + c"] \arrow[dr, Rightarrow, bend right, "\lambda k g_0 g' + c"'] & & \lambda e + c \arrow[ur, Rightarrow, "h"] \arrow[dr, Rightarrow, "k"'] & \\
 & \lambda b + c \arrow[rr, Rightarrow, "b'_1"'] \arrow[ur, dotted, "\lambda k g'_0 k' + c"] & & {}
\end{tikzcd}
$$

- **Step 1:** Additive branchings are confluent.

$$
\begin{tikzcd}
 & \lambda f'+\mu g+h \arrow[rr, Rightarrow, "a'_1"] \arrow[dr, Rightarrow, dotted, "\lambda f'+\mu b+h"] & & k \arrow[dr, Rightarrow, bend left, "a'_2"] & \\
\lambda f+\mu g+h \arrow[ur, Rightarrow, bend left, "\lambda a+\mu g+h"] \arrow[dr, Rightarrow, bend right, "\lambda f+\mu b+h"'] & & \lambda f'+\mu g'+h \arrow[ur, Rightarrow, "m"] \arrow[dr, Rightarrow, "n"'] \arrow[rr, phantom, "\text{Induction.}"] & & {} \\
 & \lambda f+\mu g'+h \arrow[rr, Rightarrow, "b'_1"'] \arrow[ur, Rightarrow, dotted, "\lambda a+\mu g'+h"'] & & k' \arrow[ur, Rightarrow, bend right, "b'_2"'] &
\end{tikzcd}
$$

- **Step 2:** Peiffer are confluent.

$$
\begin{tikzcd}
 & \lambda f'g+h \arrow[rr, Rightarrow, "a'_1"] \arrow[dr, Rightarrow, dotted, "\lambda f'b+h"] & & k \arrow[dr, Rightarrow, bend left, "a'_2"] & \\
\lambda fg+h \arrow[ur, Rightarrow, bend left, "\lambda ag+h"] \arrow[dr, Rightarrow, bend right, "\lambda fb+h"'] & & \lambda f'g'+h \arrow[ur, Rightarrow, "m"] \arrow[dr, Rightarrow, "n"'] \arrow[rr, phantom, "\text{Induction.}"] & & p \\
 & \lambda fg'+h \arrow[rr, Rightarrow, "b'_1"'] \arrow[ur, Rightarrow, dotted, "\lambda ag'+h"'] & & k' \arrow[ur, Rightarrow, bend right, "b'_2"'] &
\end{tikzcd}
$$

- **Step 3:** Overlappings are confluent.

$$
\begin{tikzcd}
 & \lambda a+c \arrow[rr, Rightarrow, "a'_1"] \arrow[dr, Rightarrow, dotted, "\lambda kf'_0k'+c"] & & l \arrow[dr, Rightarrow, bend left, "a'_2"] & \\
\lambda u+c \arrow[ur, Rightarrow, bend left, "\lambda kf_0k'+c"] \arrow[dr, Rightarrow, bend right, "\lambda kg_0g'+c"'] & & \lambda e+c \arrow[ur, Rightarrow, "h"] \arrow[dr, Rightarrow, "k"'] \arrow[rr, phantom, "\text{Induction.}"] & & p \\
 & \lambda b+c \arrow[rr, Rightarrow, "b'_1"'] \arrow[ur, Rightarrow, dotted, "\lambda kg'_0k'+c"'] & & l' \arrow[ur, Rightarrow, bend right, "b'_2"'] &
\end{tikzcd}
$$

Example: the Weyl algebras

- The **Weyl algebra** of dimension n over a field $\mathbb{K}$ is the algebra presented by the linear 2-polygraph

$$P = \langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \mid x_i x_j \Rightarrow x_j x_i, \quad \partial_i \partial_j \Rightarrow \partial_j \partial_i, \quad \partial_i x_j \Rightarrow x_j \partial_i,$$
$$\partial_i x_i \Rightarrow x_i \partial_i + 1 \text{ for any } 1 \leq i < j \leq n \rangle.$$

Example: the Weyl algebras

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- It terminates, using the degree lexicographic order on $\partial_1 > \partial_2 > \cdots > \partial_n > x_1 > x_2 > \ldots x_n$.

Example: the Weyl algebras

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- It terminates, using the degree lexicographic order on $\partial_1 > \partial_2 > \cdots > \partial_n > x_1 > x_2 > \ldots x_n$.

- It has six critical branchings:

$$\begin{tikzcd}
 & x_j x_i x_k \arrow[r, Rightarrow] & x_j x_k x_i \arrow[dr, Rightarrow] & \\
x_i x_j x_k \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & & x_k x_j x_i \\
 & x_i x_k x_j \arrow[r, Rightarrow] & x_k x_i x_j \arrow[ur, Rightarrow] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \partial_j \partial_i \partial_k \arrow[r, Rightarrow] & \partial_j \partial_k \partial_i \arrow[dr, Rightarrow] & \\
\partial_i \partial_j \partial_k \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & & \partial_k \partial_j \partial_i \\
 & \partial_i \partial_k \partial_j \arrow[r, Rightarrow] & \partial_k \partial_i \partial_j \arrow[ur, Rightarrow] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & x_j \partial_i x_k \arrow[r, Rightarrow] & x_j x_k \partial_i \arrow[dr, Rightarrow] & \\
\partial_i x_j x_k \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & & x_k x_j \partial_i \\
 & \partial_i x_k x_j \arrow[r, Rightarrow] & x_k \partial_i x_j \arrow[ur, Rightarrow] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \partial_j \partial_i x_k \arrow[r, Rightarrow] & \partial_j x_k \partial_i \arrow[dr, Rightarrow] & \\
\partial_i \partial_j x_k \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & & x_k \partial_j \partial_i \\
 & \partial_i x_k \partial_j \arrow[r, Rightarrow] & x_k \partial_i \partial_j \arrow[ur, Rightarrow] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & x_i \partial_i x_j + x_j \arrow[r, Rightarrow] & x_i x_j \partial_i + x_j \arrow[dr, Rightarrow] & \\
\partial_i x_i x_j \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & & x_j x_i \partial_i + x_j \\
 & \partial_i x_j x_i \arrow[r, Rightarrow] & x_j \partial_i x_i \arrow[ur, Rightarrow] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \partial_j \partial_i x_j \arrow[r, Rightarrow] & \partial_j x_j \partial_i \arrow[dr, Rightarrow] & \\
\partial_i \partial_j x_j \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & & x_j \partial_j \partial_i \\
 & \partial_i x_j \partial_j + \partial_i \arrow[r, Rightarrow] & x_j \partial_i \partial_j + \partial_i \arrow[ur, Rightarrow] &
\end{tikzcd}$$

- The **Weyl algebra** of dimension n over a field $\mathbb{K}$ is the algebra presented by the linear 2-polygraph

$$P = \langle x_1, \dots, x_n, \partial_1, \dots, \partial_n \mid x_i x_j \Rightarrow x_j x_i, \quad \partial_i \partial_j \Rightarrow \partial_j \partial_i, \quad \partial_i x_j \Rightarrow x_j \partial_i,$$
$$\partial_i x_i \Rightarrow x_i \partial_i + 1 \text{ for any } 1 \leq i < j \leq n \rangle.$$

- It terminates, using the degree lexicographic order on $\partial_1 > \partial_2 > \cdots > \partial_n > x_1 > x_2 > \dots x_n$.

- It has six critical branchings:

$$\begin{tikzcd}
& x_j x_i x_k \arrow[r, Rightarrow] & x_j x_k x_i \arrow[dr, Rightarrow] & \\
x_i x_j x_k \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & & x_k x_j x_i \\
& x_i x_k x_j \arrow[r, Rightarrow] & x_k x_i x_j \arrow[ur, Rightarrow] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \partial_j \partial_i \partial_k \arrow[r, Rightarrow] & \partial_j \partial_k \partial_i \arrow[dr, Rightarrow] & \\
\partial_i \partial_j \partial_k \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & & \partial_k \partial_j \partial_i \\
& \partial_i \partial_k \partial_j \arrow[r, Rightarrow] & \partial_k \partial_i \partial_j \arrow[ur, Rightarrow] &
\end{tikzcd}$$

$$\begin{tikzcd}
& x_j \partial_i x_k \arrow[r, Rightarrow] & x_j x_k \partial_i \arrow[dr, Rightarrow] & \\
\partial_i x_j x_k \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & & x_k x_j \partial_i \\
& \partial_i x_k x_j \arrow[r, Rightarrow] & x_k \partial_i x_j \arrow[ur, Rightarrow] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \partial_j \partial_i x_k \arrow[r, Rightarrow] & \partial_j x_k \partial_i \arrow[dr, Rightarrow] & \\
\partial_i \partial_j x_k \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & & x_k \partial_j \partial_i \\
& \partial_i x_k \partial_j \arrow[r, Rightarrow] & x_k \partial_i \partial_j \arrow[ur, Rightarrow] &
\end{tikzcd}$$

$$\begin{tikzcd}
& x_i \partial_i x_j + x_j \arrow[r, Rightarrow] & x_i x_j \partial_i + x_j \arrow[dr, Rightarrow] & \\
\partial_i x_i x_j \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & & x_j x_i \partial_i + x_j \\
& \partial_i x_j x_i \arrow[r, Rightarrow] & x_j \partial_i x_i \arrow[ur, Rightarrow] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \partial_j \partial_i x_j \arrow[r, Rightarrow] & \partial_j x_j \partial_i \arrow[dr, Rightarrow] & \\
\partial_i \partial_j x_j \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & & x_j \partial_j \partial_i \\
& \partial_i x_j \partial_j + \partial_i \arrow[r, Rightarrow] & x_j \partial_i \partial_j + \partial_i \arrow[ur, Rightarrow] &
\end{tikzcd}$$

- A linear basis of the Weyl algebra of dimension n is given by the elements $x_n^{\alpha_n} \dots x_1^{\alpha_1} \partial_n^{\beta_n} \dots \partial_1^{\beta_1}$ for $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \in \mathbb{N}$.

III. Squier's coherence theorem

- A **linear 3-polygraph** is a quadruple (P_0, P_1, P_2, P_3) made of
  - a linear 2-polygraph (P_0, P_1, P_2),
  - a cellular extension $P_3 \overset{s_2}{\underset{t_2}{\rightrightarrows}} P_2^{\ell}$ satisfying globular condition:

$$\begin{tikzcd} s_0(A) \arrow[rr, bend left=60, "s_1(A)"] \arrow[rr, bend right=60, "t_1(A)"'] & s_2(A)\Downarrow \overset{A}{\Rrightarrow} \Downarrow t_2(A) & t_0(A) \end{tikzcd}$$

- A **linear 3-polygraph** is a quadruple (P_0, P_1, P_2, P_3) made of
  - a linear 2-polygraph (P_0, P_1, P_2),
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$$\begin{tikzcd} s_0(A) \arrow[rr, bend left, "s_1(A)"] \arrow[rr, bend right, "t_1(A)"'] \arrow[rr, phantom, "s_2(A)\Downarrow \overset{A}{\Rrightarrow} \Downarrow t_2(A)"] & & t_0(A) \end{tikzcd}$$

- A **3-algebra** is given by the data of a diagram in **Alg**:

$$\begin{tikzcd} A_1 \arrow[r, bend left, "i_2"] & A_2 \arrow[l, shift right, "s_1"'] \arrow[l, shift left, "t_1"] \arrow[r, bend left, "i_3"] & A_3 \arrow[l, shift right, "s_2"'] \arrow[l, shift left, "t_2"] & \\ & A_2 \times_{A_1} A_2 \arrow[u, "\star_1"] & A_3 \times_{A_2} A_3 \arrow[u, "\star_2"] & A_3 \times_{A_1} A_3 \arrow[ul, "\star_1"'] \end{tikzcd}$$

- One defines the free 3-algebra P_3^ℓ generated by a linear 3-polygraph P.

Homotopy bases and coherent confluence

- A **linear 3-polygraph** is a quadruple (P_0, P_1, P_2, P_3) made of
  - a linear 2-polygraph (P_0, P_1, P_2),
  - a cellular extension $P_3 \underset{t_2}{\overset{s_2}{\rightrightarrows}} P_2^\ell$ satisfying globular condition:

$$\begin{tikzcd} s_0(A) \arrow[rr, bend left=50, "s_1(A)"] \arrow[rr, bend right=50, "t_1(A)"'] \arrow[rr, phantom, "s_2(A)\Downarrow \overset{A}{\Rrightarrow} \Downarrow t_2(A)"] & & t_0(A) \end{tikzcd}$$

- A **3-algebra** is given by the data of a diagram in **Alg**:

$$\begin{tikzcd}
A_1 \arrow[r, bend left, "i_2"] & A_2 \arrow[l, shift right, "s_1"'] \arrow[l, shift left, "t_1"] \arrow[r, bend left, "i_3"] & A_3 \arrow[l, shift right, "s_2"'] \arrow[l, shift left, "t_2"] & \\
 & A_2 \times_{A_1} A_2 \arrow[u, "\star_1"] & A_3 \times_{A_2} A_3 \arrow[u, "\star_2"] & A_3 \times_{A_1} A_3 \arrow[ul, "\star_1"']
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Homotopy bases and coherent confluence

- A **linear 3-polygraph** is a quadruple (P_0, P_1, P_2, P_3) made of
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$$\begin{tikzcd}[column sep=large] s_0(A) \arrow[rr, bend left=50, "s_1(A)"] \arrow[rr, bend right=50, "t_1(A)"'] & {\scriptstyle s_2(A)}\Downarrow \overset{A}{\Rrightarrow} \Downarrow{\scriptstyle t_2(A)} & t_0(A) \end{tikzcd}$$

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- **Theorem (Coherent Newman lemma):** If P is terminating, then P is P_3-confluent if and only if P is locally P_3-confluent.

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f \arrow[r, Rightarrow, "\beta_1"] \arrow[d, Rightarrow, "\eta_f"'] & g_1 \arrow[r, Rightarrow, "\gamma_1^-"] \arrow[d, Rightarrow, "\eta_{g_1}"] \arrow[dl, Rrightarrow, "\eta_{\beta_1}"] & f_2 \arrow[r, Rightarrow] \arrow[d, Rightarrow, "\eta_{f_2}"] \arrow[dl, Rrightarrow, "\eta_{\gamma_1^-}"] & \cdots \arrow[r, Rightarrow] & f_p \arrow[r, Rightarrow, "g_p"] \arrow[d, Rightarrow, "\eta_{f_p}"] & g_p \arrow[r, Rightarrow, "h_p^-"] \arrow[d, Rightarrow, "\eta_{b_p}"] \arrow[dl, Rrightarrow, "\eta_{\beta_p}"] & g \arrow[d, Rightarrow, "\eta_g"] \arrow[dl, Rrightarrow, "\eta_{\gamma_p^-}"] \\
\overline{f} \arrow[r, equal] & \overline{f} \arrow[r, equal] & \overline{f} \arrow[r, equal] & \cdots \arrow[r, equal] & \overline{f} \arrow[r, equal] & \overline{f} \arrow[r, equal] & \overline{f}
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  - Finally, for all parallel 2-cells $\alpha, \beta : f \to g$ of P_2^ℓ, the composite 3-cell

$$\begin{tikzcd} & g \arrow[dr, Rightarrow, "\eta_g"] \arrow[drr, Rightarrow, bend left, "1_g"] \arrow[d, phantom, "\Downarrow\eta_\alpha"] & & \\ f \arrow[ur, Rightarrow, "\alpha"] \arrow[dr, Rightarrow, "\beta"'] \arrow[rr, Rightarrow, "\eta_f"] & {} & \overline{f} = \eta_g^- \arrow[r, Rightarrow] & g \\ & g \arrow[ur, Rightarrow, "\eta_g"'] \arrow[urr, Rightarrow, bend right, "1_g"'] \arrow[u, phantom, "\Downarrow\eta_\beta^-"] & & \end{tikzcd}$$

Squier's theorem

- **Squier's coherence theorem:** Let P be a convergent left-monomial linear 2-polygraph. A cellular extension P_3 of P_2^{ℓ} that contains a 3-cell

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$$f \overset{\alpha}{\Rightarrow} g \overset{\alpha'}{\Rightarrow} k, \quad f \overset{\beta}{\Rightarrow} h \overset{\beta'}{\Rightarrow} k, \quad F : \alpha \star_1 \alpha' \Rrightarrow \beta \star_1 \beta'$$

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 & -x^4 & \\
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 & \mu yx^2z \arrow[ur, Rightarrow, "\mu\gamma z"'] &
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zyx^2 \arrow[ur, Rightarrow, "\beta x^2"] \arrow[dr, Rightarrow, "z\gamma"'] & & x^2zy \arrow[ul, Rightarrow, "x^2\beta"'] \\
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Squier's theorem

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$$\begin{tikzcd} & g \arrow[dr, Rightarrow, "\alpha'"] \arrow[d, phantom, "\Downarrow F"] & \\ f \arrow[ur, Rightarrow, "\alpha"] \arrow[dr, Rightarrow, "\beta"'] & {} & k \\ & h \arrow[ur, Rightarrow, "\beta'"'] & \end{tikzcd}$$

for every critical branching (α, β) of P, with α' and β' positive 2-cells of P_2^ℓ, is acyclic.

- **Example:** Consider $P = \langle x, y, z \mid yz \overset{\alpha}{\Rightarrow} -x^2,\ zy \overset{\beta}{\Rightarrow} -\mu x^2 \rangle$ presenting the algebra $A = \mathbb{K}\langle x, y, z \mid x^2 + yz = 0,\ x^2 + \lambda zy = 0 \rangle$, with $\mu := \lambda^{-1}$.
  - It terminates, using the deglex order generated by $z > y > x$.
  - Squier's completion:

$$\begin{tikzcd} & -x^2y \arrow[d, phantom, "\Downarrow A"] \\ yzy \arrow[ur, Rightarrow, "\alpha y"] \arrow[dr, Rightarrow, "y\beta"'] & {} \\ & -\mu yx^2 \arrow[uu, Rightarrow, bend right, "-\mu\gamma"'] \end{tikzcd} \qquad \begin{tikzcd} & -\mu x^2z \arrow[d, phantom, "\Downarrow B"] \\ zyz \arrow[ur, Rightarrow, "\beta z"] \arrow[dr, Rightarrow, "z\alpha"'] & {} \\ & -zx^2 \arrow[uu, Rightarrow, bend right, "-\delta"'] \end{tikzcd}$$

$$\begin{tikzcd} & -x^4 \arrow[d, phantom, "\Downarrow C"] & \\ yzx^2 \arrow[ur, Rightarrow, "\alpha x^2"] \arrow[dr, Rightarrow, "y\delta"'] & {} & x^2yz \arrow[ul, Rightarrow, "x^2\alpha"'] \\ & \mu yx^2z \arrow[ur, Rightarrow, "\mu\gamma z"'] & \end{tikzcd} \qquad \begin{tikzcd} & -\mu x^4 \arrow[d, phantom, "\Downarrow D"] & \\ zyx^2 \arrow[ur, Rightarrow, "\beta x^2"] \arrow[dr, Rightarrow, "z\gamma"'] & {} & x^2zy \arrow[ul, Rightarrow, "x^2\beta"'] \\ & \lambda zx^2y \arrow[ur, Rightarrow, "\lambda\delta y"'] & \end{tikzcd}$$

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  - $\langle x, y, z \mid \alpha, \beta, \gamma, \delta \mid A, B, C, D \rangle$ is a coherent presentation of A.

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- **Example:** The linear 2-polygraph $P = \langle x, y, z \mid xyz \Rightarrow x^3 + y^3 + z^3 \rangle$ is convergent and admits an empty homotopy basis.