Homotopical motivations of higher dimensinal rewriting

Maxime Lucas
LIPN, Université Sorbonne Paris Nord

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Outline

- From monoid actions to weak actions
- From weak actions to coherent presentations
- A hands on introduction to model category structures
- Link with homological algebra

Monoid action

Take a monoid M acting on a groupoid $\mathcal{C}$.

- For any $m \in M$, $\bar{m} : \mathcal{C} \to \mathcal{C}$,

$$\bar{m} \circ \bar{n} = \overline{mn} \qquad \bar{1} = id$$

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$$\begin{tikzcd} \bar{m} \arrow[r, bend right] & \mathcal{C} \arrow[rr, bend left, "F"] \arrow[rr, phantom, "\simeq"] & & \mathcal{D} \arrow[ll, bend left, "G"] \end{tikzcd}$$

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- Define $\bar{m}(x) := F(\bar{m}(G(x)))$: this is *not* a monoid action!

First solution: weak actions

$$\begin{tikzcd} \bar{m} \arrow[r, hook] & \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\simeq"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

On $\mathcal{D}$, we only get natural isomorphisms:

$$\alpha^{m,n} : \bar{m} \circ \bar{n} \simeq \overline{mn} \qquad \beta : \bar{1} \simeq id$$

Those satisfy the following equation (+ equations involving β):

$$\begin{tikzcd} & \overline{m_1 m_2} \circ \bar{m}_3 \arrow[dr, "\alpha^{m_1 m_2, m_3}"] & \\ \bar{m}_1 \circ \bar{m}_2 \circ \bar{m}_3 \arrow[ur, "\alpha^{m_1, m_2} \circ \bar{m}_3"] \arrow[dr, "\bar{m}_1 \circ \alpha^{m_2, m_3}"'] & & \overline{m_1 m_2 m_3} \\ & \bar{m}_1 \circ \overline{m_2 m_3} \arrow[ur, "\alpha^{m_1, m_2 m_3}"'] & \end{tikzcd}$$

2nd solution: the case of free monoids

$$M = E^*$$

2nd solution: the case of presented monoids

$$M = \langle a | aa = 1 \rangle$$

Summary

- An action of M is replaced by an action of $\bar{\Sigma}$, where Σ is a coherent presentation of M.
- Weak actions correspond to (strict) actions by the standard coherent presentation fo M.

A few questions:

- In what way are coherent presentations "equivalent"?
- What is the relationship between coherent presentations and M?
- Goal: Push all this to higher dimensions...

Historically, this is better understood in the case where M is an operad, which acts on topological spaces. Axiomatized using model category structures.

Lifting properties

Definition

$f, g \in \mathcal{C}$.
We say that f is left-orthogonal to g (or g is right orthogonal to f) if for all i, j, there exists h such that:

Given a set of arrows X, we denote by X^r (resp. X^l) the set of its right (resp. left) orthogonals.

Examples

- Injective maps
- Surjective maps
- Essentially surjective functor
- Faithful funtor
- Fully faithful functor.

Proposition

- *X^r is closed under pullbacks and retracts*
- *X^l is closed under pushouts, retracts and transfinite composition.*
- $X^{rlr} = X^r$, $X^{lrl} = X^l$.

Proposition (Small object argument)

In good cases:

- *X^{rl} is the (closure under retract of) transfinite compositions of pushouts of arrows of X.*
- *Any arrow $f : u \to v$ factorises as an arrow of X^{rl} followed by an arrow of X^l.*

The case of ω-categories

$$X = \{\iota_n : \partial D_n \to D_n\}$$

X^r:

X^{rl}:

Cofibrant replacements

In our case, we call elements of X^{lr} the *cofibrations* and of X^{r} the *trivial cofibrations*.
An ω-category C is "free on a palygraph" iff the arrow $\emptyset \to C$ is a cofibration (we also say that C is cofibrant)
We say that $\bar{C}$ it is a *cofibrant replacement* of C if it is cofibrant and $\bar{C} \to C$ is a trivial fibration.
We are just factorizing the map $\emptyset \to C$!

Model category structure

Definition

A model categroy structure on a complete, cocomplete category $\mathcal{C}$ is the data of three classes of maps: W, C and F such that:

- W has the 2-out-of-3 property
- $WC = F^l$, and $WF = C^r$.
- Any arrow factors as a map in C (resp. CW) followed by a map in FW (resp. F).