Identities and bases in hypoplactic, sylvester, Baxter and stylic monoids

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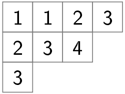
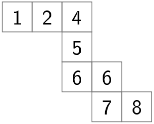
Plactic-like monoids

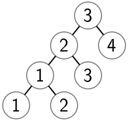
Plactic monoid
plac
Young tableau



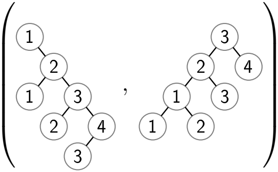
Hypoplactic monoid
hypo
Quasi-ribbon tableau



Sylvester monoid
sylv
Right strict binary search tree



Baxter monoid
baxt
Pair of twin binary search trees



- Elements of each monoid can be identified with combinatorial objects of a particular type;
- Insertion algorithms allows one to compute unique combinatorial objects for each word over an ordered alphabet;
- Each monoid arises from combinatorial Hopf algebras, whose bases are indexed by combinatorial objects (Giraudo 2012; Hivert et al. 2005; Krob and Thibon 1997).

Questions we wanted to answer:

1. Do plactic-like monoids of higher rank embed into their corresponding monoids of lesser rank, or direct products of the latter?
2. What are the varieties generated by plactic-like monoids of rank greater than or equal to 2?
3. Can we obtain a characterization of the identities satisfied by plactic-like monoids?
4. Can we obtain a finite basis for the varieties generated by the plactic-like monoids? And can we obtain their axiomatic rank?

Let $A := \{1 < 2 < \cdots\}$ be the infinite ordered alphabet and let $A_n = \{1 < 2 < \cdots < n\}$ be the ordered alphabet with n letters.

The plactic monoid

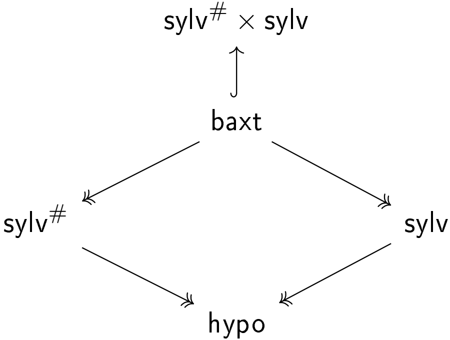
Young tableau

1	1	2	3	5	7
3	4	6			
5	5	7			
6					

- Rows weakly increasing left to right;
- Columns strictly increasing top to bottom;
- Left-justified, longer rows on top.

- Schensted's algorithm computes a unique tableau $\mathrm{P}_{\mathrm{plac}}^{\rightarrow}(u)$ for each word $u \in A^*$.
- Define the congruence $\equiv_{\mathrm{plac}}$ on A^* as follows: For $u, v \in A^*$,
$$u \equiv_{\mathrm{plac}} v \iff \mathrm{P}_{\mathrm{plac}}^{\rightarrow}(u) = \mathrm{P}_{\mathrm{plac}}^{\rightarrow}(v).$$
- The factor monoid $\mathrm{plac} := A^*/{\equiv_{\mathrm{plac}}}$ is the infinite-rank plactic monoid (Lascoux and Schützenberger 1981). Analogously, $\mathrm{plac}_n := A_n^*/{\equiv_{\mathrm{plac}}}$ is the plactic monoid of finite rank n.

The following diagram commutes:



Identities and varieties

- A monoid identity, over an alphabet of variables X, is a formal equality $u \approx v$, where u and v are words in the free monoid X^*.
- A monoid M satisfies the identity $u \approx v$ if equality in M holds under every substitution of the variables of u and v by elements of M.
- The set of all identities satisfied by a monoid is its equational theory.
- An equational class of monoids is the class of all monoids which satisfy the same equational theory.
- A variety of monoids is a nonempty class of monoids closed under submonoids, homomorphic images and direct products.
- A nonempty class of monoids is a variety if and only if it is an equational class (Birkhoff 1935).

Identities and plactic-like monoids

Known facts about identities and the plactic monoids:

- plac_2 satisfies exactly the same identities as the monoid of 2×2 upper triangular tropical matrices (Izhakian 2019) and the bicyclic monoid (Daviaud et al. 2018), hence, it satisfies Adian's identity

$$xyyxxyxyyx \approx xyyxyxxyyx.$$

- plac_3 satisfies the identity

$$uvvuvu \approx uvuvvu,$$

where $u(x, y)$ and $v(x, y)$ are respectively the left and right side of Adian's identity, but does not satisfy Adian's identity itself (Kubat and Okniński 2015).

- plac_n does not satisfy any non-trivial identity of length less than or equal to n, hence, plac does not satisfy any non-trivial identity (Cain et al. 2017).

- Monoids of upper triangular matrices over the tropical semiring satisfy non-trivial identities (Izhakian 2014; Okniński 2015; Taylor 2017).
- Recently, a faithful tropical representation for plac_n was given by Marianne Johnson and Kambites (2021), thus showing that plac_n satisfies a non-trivial identity.
- More recently, Cain et al. (2022a) gave faithful representations of the hypoplactic, stalactic, taiga, sylvester, Baxter and right patience sorting monoids of each finite rank as monoids of upper triangular matrices over certain semirings.

The shortest non-trivial identities satisfied by the hypoplactic, sylvester and Baxter monoids have been characterized by Cain and Malheiro (2018):

Monoid	Shortest identities
hypo	$xyxy \approx xyyx \approx yxxy \approx yxyx$ $xxyx \approx xyxx$
sylv	$xyxy \approx yxxy$
$\text{sylv}^{\#}$	$yxyx \approx yxxy$
baxt	$yxxyxy \approx yxyxxy$ $xyxyxy \approx xyyxxy$

Bases and axiomatic rank

- If every identity in a set of identities Σ is a consequence of a subset $\mathcal{B}$ of Σ, then $\mathcal{B}$ is a basis of Σ.
- The axiomatic rank of Σ is the least natural number r such that Σ admits a basis $\mathcal{B}$ where the number of distinct variables occurring in each identity in $\mathcal{B}$ does not exceed r.
- If a variety $\mathbf{\mathcal{V}}$ is generated by a monoid with a finite number of generators, the minimal such number is called the basis rank of $\mathbf{\mathcal{V}}$.

The variety generated by the bicyclic monoid has infinite axiomatic rank (Shneerson 1989). By Daviaud et al. 2018, Theorem 4.1, this implies that $\mathbf{\mathcal{V}}_{\mathsf{plac}_2}$ has infinite axiomatic rank.

The hypoplactic monoid

Quasi-ribbon tableau

1	2	4		
		5		
		6	6	
			7	8

- Rows weakly increasing left to right;
- Columns strictly increasing top to bottom;
- Leftmost cell in each row is below the rightmost cell of the previous row.

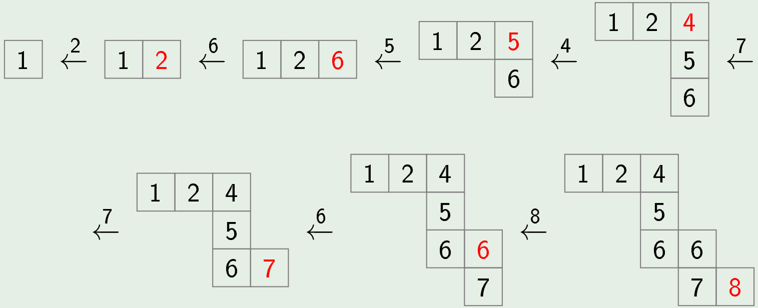
- The Krob–Thibon algorithm computes a unique quasi-ribbon tableau $\mathrm{P}_{\mathrm{hypo}}^{\rightarrow}(u)$ for each word $u \in A^*$.
- Define the congruence $\equiv_{\mathrm{hypo}}$ on A^* as follows: For $u, v \in A^*$,

$$u \equiv_{\mathrm{hypo}} v \iff \mathrm{P}_{\mathrm{hypo}}^{\rightarrow}(u) = \mathrm{P}_{\mathrm{hypo}}^{\rightarrow}(v).$$

- The factor monoid $\mathrm{hypo} := A^*/{\equiv_{\mathrm{hypo}}}$ is the infinite-rank hypoplactic monoid (Krob and Thibon 1997). Analogously, $\mathrm{hypo}_n := A_n^*/{\equiv_{\mathrm{hypo}}}$ is the hypoplactic monoid of finite rank n.

Example (Krob–Thibon algorithm)

Computing $\mathrm{P}^{\rightarrow}_{\mathrm{hypo}}(12654768)$:



A consequence of the Krob–Thibon insertion algorithm:

Corollary (Novelli 2000)

Let $u, v \in A^*$. Then, $u \equiv_{\mathrm{hypo}} v$ if and only if u and v share the same content and inversions.

A word u with $\mathrm{supp}(u) = \{a_1 < \cdots < a_k\}$ has an a_{i+1}-a_i inversion if, when reading u from left to right, a_{i+1} occurs before a_i.

Example

- 31214 has 3-2 and 2-1 inversions;
- 21341 has a 2-1 inversion;
- 724 has a 7-4 inversion.

Embeddings of the hypoplactic monoids

For any $i, j \in A$, with $i < j$, define a map from A to hypo_2 in the following way: For any $a \in A$,

$$a \longmapsto \begin{cases} [1]_{\mathrm{hypo}_2} & \text{if } a = i; \\ [2]_{\mathrm{hypo}_2} & \text{if } a = j; \\ [21]_{\mathrm{hypo}_2} & \text{if } i < a < j; \\ [\varepsilon]_{\mathrm{hypo}_2} & \text{otherwise;} \end{cases}$$

and extend it to a homomorphism from A^* to hypo_2, which factors to give a homomorphism $\varphi^{ij}_{\mathrm{hypo}} : \mathrm{hypo} \longrightarrow \mathrm{hypo}_2$.

Lemma (Cain, Malheiro, and R. 2022)

Let $u, v \in A_n^*$. Then, $u \equiv_{\mathrm{hypo}} v$ if and only if

$$\varphi^{ij}_{\mathrm{hypo}}([u]_{\mathrm{hypo}}) = \varphi^{ij}_{\mathrm{hypo}}([v]_{\mathrm{hypo}}),$$

for all $1 \leq i < j \leq n$.

Example

$$[31214]_{\mathrm{hypo}} \underset{\varphi_{12}}{\longmapsto} [121]_{\mathrm{hypo}_2}$$

$$[31214]_{\mathrm{hypo}} \underset{\varphi_{13}}{\longmapsto} [21211]_{\mathrm{hypo}_2}$$

$$[31214]_{\mathrm{hypo}} \underset{\varphi_{14}}{\longmapsto} [2112112]_{\mathrm{hypo}_2}$$

$$[31214]_{\mathrm{hypo}} \underset{\varphi_{23}}{\longmapsto} [21]_{\mathrm{hypo}_2}$$

$$[31214]_{\mathrm{hypo}} \underset{\varphi_{24}}{\longmapsto} [2112]_{\mathrm{hypo}_2}$$

$$[31214]_{\mathrm{hypo}} \underset{\varphi_{34}}{\longmapsto} [12]_{\mathrm{hypo}_2}$$

Example

$$[21341]_{\mathsf{hypo}} \underset{\varphi_{12}}{\longmapsto} [211]_{\mathsf{hypo}_2}$$

$$[21341]_{\mathsf{hypo}} \underset{\varphi_{13}}{\longmapsto} [21121]_{\mathsf{hypo}_2}$$

$$[21341]_{\mathsf{hypo}} \underset{\varphi_{14}}{\longmapsto} [2112121]_{\mathsf{hypo}_2}$$

$$[21341]_{\mathsf{hypo}} \underset{\varphi_{23}}{\longmapsto} [12]_{\mathsf{hypo}_2}$$

$$[21341]_{\mathsf{hypo}} \underset{\varphi_{24}}{\longmapsto} [1212]_{\mathsf{hypo}_2}$$

$$[21341]_{\mathsf{hypo}} \underset{\varphi_{34}}{\longmapsto} [12]_{\mathsf{hypo}_2}$$

Example

$$[724]_{\mathsf{hypo}} \underset{\varphi_{24}}{\longmapsto} [12]_{\mathsf{hypo}_2}$$

$$[724]_{\mathsf{hypo}} \underset{\varphi_{27}}{\longmapsto} [2121]_{\mathsf{hypo}_2}$$

$$[724]_{\mathsf{hypo}} \underset{\varphi_{47}}{\longmapsto} {\color{red}[21]_{\mathsf{hypo}_2}}$$

$$[724]_{\mathsf{hypo}} \underset{\varphi_{23}}{\longmapsto} [1]_{\mathsf{hypo}_2}$$

$$[724]_{\mathsf{hypo}} \underset{\varphi_{56}}{\longmapsto} [\varepsilon]_{\mathsf{hypo}_2}$$

For each $n \geq 3$, let $I_n := \{(i,j) : 1 \leq i < j \leq n\}$, and consider the map

$$\phi_{\mathsf{hypo}_n} : \mathsf{hypo}_n \longrightarrow \prod_{I_n} \mathsf{hypo}_2,$$

whose (i,j)-th component is given by $\varphi^{ij}_{\mathsf{hypo}}([w]_{\mathsf{hypo}})$, for $w \in A_n^*$ and $(i,j) \in I_n$.

Proposition (Cain, Malheiro, and R. 2022)

The map ϕ_{hypo_n} is an embedding.

Theorem (Cain, Malheiro, and R. 2022)

For any $n \geq 2$, hypo and hypo_n satisfy exactly the same identities.

Proposition (Cain, Malheiro, and R. 2022)

The basis rank of $\mathcal{V}_{\mathsf{hypo}}$ is 2.

Identities satisfied by hypo

Theorem (Cain, Malheiro, and R. 2022)

The identities $u \approx v$ satisfied by hypo *are those balanced identities such that, for any variables x, y that occur in u and v, u admits xy as a subsequence if and only if v does too.*

Corollary (Cain, Malheiro, and R. 2022)

$\mathcal{V}_{\mathsf{hypo}}$ *is the varietal join $\mathbf{C} \vee \mathcal{V}(\mathcal{C}_3)$, and is generated by the free monogenic monoid and the monoid $\mathcal{C}_3$, the 5-element monoid of all order preserving and extensive transformations of the chain*

$$1 < 2 < 3.$$

Corollary (Cain, Malheiro, and R. 2022)

The decision problem Check-Id(hypo) *belongs to the complexity class* P.

Corollary (Cain, Malheiro, and R. 2022)

The shortest non-trivial identity, with n variables, satisfied by hypo*, is of length $n+2$.*

Example

The identity

$$xa_1 \ldots a_{n-1}{\color{red}x}x \approx x{\color{red}x}a_1 \ldots a_{n-1}x$$

is a minimum-length non-trivial identity, with n variables, satisfied by hypo.

Finite basis and axiomatic rank of $\mathcal{V}_{\text{hypo}}$

Theorem (Cain, Malheiro, and R. 2022)

$\mathcal{V}_{\text{hypo}}$ admits a finite basis, consisting of the following identities:

$$xyzxty \approx yxzxty; \tag{L}$$

$$xzxytx \approx xzyxtx; \tag{M}$$

$$xzytxy \approx xzytyx. \tag{R}$$

Proposition (Cain, Malheiro, and R. 2022)

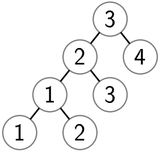
Neither of the identities (L) *or* (R) *is a consequence of the set of non-trivial identities, satisfied by* hypo, *over an alphabet with four variables, excluding itself (but not the other) and equivalent identities.*

Corollary (Cain, Malheiro, and R. 2022)

The axiomatic rank of $\mathcal{V}_{\text{hypo}}$ is 4.

The sylvester monoid

Right strict binary search tree



Label of each node is:

- greater than or equal to the label of every node in its left subtree;
- strictly less than the label of every node in its right subtree.

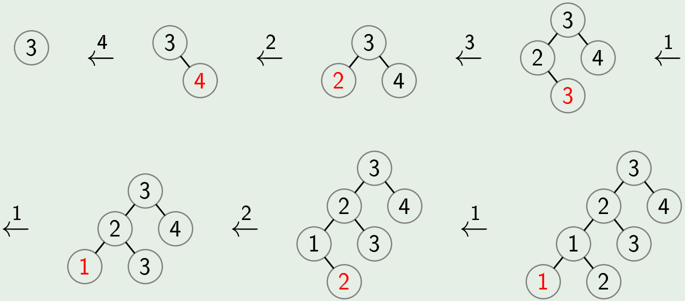
- The Hivert–Novelli–Thibon algorithm computes a unique right strict binary search tree $\mathrm{P}_{\mathsf{sylv}}^{\leftarrow}(u)$ for each word $u \in A^*$.
- Define the congruence $\equiv_{\mathsf{sylv}}$ on A^* as follows: For $u, v \in A^*$,

$$u \equiv_{\mathsf{sylv}} v \iff \mathrm{P}_{\mathsf{sylv}}^{\leftarrow}(u) = \mathrm{P}_{\mathsf{sylv}}^{\leftarrow}(v).$$

- The factor monoid $\mathsf{sylv} := A^*/{\equiv_{\mathsf{sylv}}}$ is the infinite-rank sylvester monoid (Hivert et al. 2005). Analogously, $\mathsf{sylv}_n := A_n^*/{\equiv_{\mathsf{sylv}}}$ is the sylvester monoid of finite rank n.

Example (Hivert–Novelli–Thibon algorithm)

Computing $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(1213243)$:



A consequence of the Hivert–Novelli–Thibon insertion algorithm:

Proposition (Cain, Malheiro, and R. 2021)

Let $u, v \in A^*$. Then, $u \equiv_{\mathsf{sylv}} v$ if and only if u and v share exactly the same content and right precedences.

A word u has a b-a right precedence if, when reading u from right to left, b is the least element greater than a which occurs before the first occurrence of a, for $a, b \in \operatorname{supp}(u)$. The number of occurrences of b before the first occurrence of a is the index of the right precedence.

Example

- 3123 has a 2-1 and a 3-2 right precedence, both of index 1;
- 2313 has a 3-1 right precedence of index 1 and a 3-2 right precedence of index 2;
- 3132 has a 2-1 right precedence of index 1.

Embeddings of the sylvester monoids

For any $i, j \in A$, with $i < j$, define a map from A to sylv_2 in the following way: For any $a \in A$,

$$a \longmapsto \begin{cases} [1]_{\mathrm{sylv}_2} & \text{if } a = i; \\ [2]_{\mathrm{sylv}_2} & \text{if } a = j; \\ [21]_{\mathrm{sylv}_2} & \text{if } i < a < j; \\ [\varepsilon]_{\mathrm{sylv}_2} & \text{otherwise}; \end{cases}$$

and extend it to a homomorphism from A^* to sylv_2, which factors to give a homomorphism $\varphi^{ij}_{\mathrm{sylv}} : \mathrm{sylv} \longrightarrow \mathrm{sylv}_2$.

Lemma (Cain, Malheiro, and R. 2021)

Let $u, v \in A_n^*$. Then, $u \equiv_{\mathrm{sylv}} v$ if and only if

$$\varphi^{ij}_{\mathrm{sylv}}([u]_{\mathrm{sylv}}) = \varphi^{ij}_{\mathrm{sylv}}([v]_{\mathrm{sylv}}),$$

for all $1 \leq i < j \leq n$.

Example

$$[3123]_{\mathsf{sylv}_3} \underset{\varphi^{12}_{\mathsf{sylv}}}{\longmapsto} [12]_{\mathsf{sylv}_2}$$

$$[3123]_{\mathsf{sylv}_3} \underset{\varphi^{13}_{\mathsf{sylv}}}{\longmapsto} [21212]_{\mathsf{sylv}_2}$$

$$[3123]_{\mathsf{sylv}_3} \underset{\varphi^{23}_{\mathsf{sylv}}}{\longmapsto} [212]_{\mathsf{sylv}_2}$$

Example

$$[2313]_{\mathsf{sylv}_3} \underset{\varphi^{12}_{\mathsf{sylv}}}{\longmapsto} [21]_{\mathsf{sylv}_2}$$

$$[2313]_{\mathsf{sylv}_3} \underset{\varphi^{13}_{\mathsf{sylv}}}{\longmapsto} [21212]_{\mathsf{sylv}_2}$$

$$[2313]_{\mathsf{sylv}_3} \underset{\varphi^{23}_{\mathsf{sylv}}}{\longmapsto} [122]_{\mathsf{sylv}_2}$$

For each $n \geq 3$, let $I_n := \{(i,j) : 1 \leq i < j \leq n\}$, and consider the map

$$\phi_{\mathrm{sylv}_n} : \mathrm{sylv}_n \longrightarrow \prod_{I_n} \mathrm{sylv}_2,$$

whose (i,j)-th component is given by $\varphi^{ij}_{\mathrm{sylv}}([w]_{\mathrm{sylv}})$, for $w \in A_n^*$ and $(i,j) \in I_n$.

Proposition (Cain, Malheiro, and R. 2021)

The map ϕ_{sylv_n} is an embedding.

Theorem (Cain, Malheiro, and R. 2021)

For any $n \geq 2$, sylv and sylv_n satisfy exactly the same identities.

Proposition (Cain, Malheiro, and R. 2021)

The basis rank of $\mathcal{V}_{\mathrm{sylv}}$ is 2.

Identities satisfied by sylv

Theorem (Cain, Malheiro, and R. 2021)

The identities $u \approx v$ *satisfied by* sylv *are those balanced identities such that, for any* $x \in \operatorname{supp}(u \approx v)$*, the longest suffix of* u *where* x *does not occur has the same content as the longest suffix of* v *where* x *does not occur.*

Corollary (Cain, Malheiro, and R. 2021)

The decision problem Check-Id(sylv) *belongs to the complexity class* P.

Corollary (Cain, Malheiro, and R. 2021)

The shortest non-trivial identity, with n variables, satisfied by sylv, *is of length $n + 2$.*

Example

The identity

$$xy a_1 \dots a_{n-2} yx \approx yx a_1 \dots a_{n-2} yx$$

is a minimum-length non-trivial identity, with n variables, satisfied by sylv.

Finite basis and axiomatic rank of $\mathcal{V}_{\mathsf{sylv}}$

Theorem (Cain, Malheiro, and R. 2021)

$\mathcal{V}_{\mathsf{sylv}}$ admits a finite basis, consisting of the following identity:

$$xyzxty \approx yxzxty. \tag{L}$$

Proposition (Cain, Malheiro, and R. 2021)

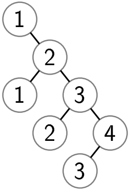
The identity (L) is not a consequence of the set of non-trivial identities, satisfied by sylv*, over an alphabet with four variables, excluding itself and equivalent identities.*

Corollary (Cain, Malheiro, and R. 2021)

The axiomatic rank of $\mathcal{V}_{\mathsf{sylv}}$ is 4*.*

The #-sylvester monoid

Left strict binary search tree



Label of each node is:

- strictly greater than the label of every node in its left subtree;
- less than or equal to the label of every node in its right subtree.

- An analogue of the Hivert–Novelli–Thibon algorithm computes a unique left strict binary search tree $\mathrm{P}^{\rightarrow}_{\mathsf{sylv}^{\#}}(u)$ for each word $u \in A^*$.
- Define the congruence $\equiv_{\mathsf{sylv}^{\#}}$ on A^* as follows: For $u, v \in A^*$,

$$u \equiv_{\mathsf{sylv}^{\#}} v \iff \mathrm{P}^{\rightarrow}_{\mathsf{sylv}^{\#}}(u) = \mathrm{P}^{\rightarrow}_{\mathsf{sylv}^{\#}}(v).$$

- The factor monoid $\mathrm{sylv}^{\#} := A^*/{\equiv_{\mathrm{sylv}^{\#}}}$ is the infinite-rank #-sylvester monoid. Analogously, $\mathrm{sylv}^{\#}_n := A^*/{\equiv_{\mathrm{sylv}^{\#}_n}}$ is the #-sylvester monoid of finite rank n.
- For each $n \in \mathbb{N}$, sylv_n is anti-isomorphic to $\mathrm{sylv}^{\#}_n$.

Proposition (Cain, Malheiro, and R. 2021)

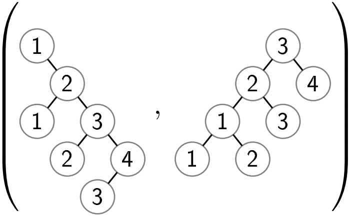
There is no anti-isomorphism between sylv *and* $\mathrm{sylv}^{\#}$.

Proposition (Cain, Malheiro, and R. 2021)

Let $u, v \in A^*$. Then, $u \equiv_{\mathrm{sylv}^{\#}} v$ if and only if u and v share exactly the same content and left precedences.

The Baxter monoid

Pair of twin binary search trees



- Consists of a left strict binary search tree and a right strict binary search tree;
- The trees have the same content and their canopies are complementary.

- For each $u \in A^*$, the pair of binary search trees

$$\left(\mathrm{P}^{\rightarrow}_{\mathsf{sylv}^\#}(u), \mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(u)\right)$$

is a pair of twin binary search trees (Giraudo 2012, Proposition 4.5), denoted by $\mathrm{P}_{\mathsf{baxt}}(u)$.

- Define the congruence $\equiv_{\text{baxt}}$ on A^* as follows: For $u, v \in A^*$,

$$u \equiv_{\text{baxt}} v \iff \mathrm{P}_{\text{baxt}}(u) = \mathrm{P}_{\text{baxt}}(v).$$

- The factor monoid $\text{baxt} := A^*/{\equiv_{\text{baxt}}}$ is the infinite-rank Baxter monoid (Giraudo 2012). Analogously, $\text{baxt}_n := A_n^*/{\equiv_{\text{baxt}}}$ is the Baxter monoid of finite rank n.

Corollary (Cain, Malheiro, and R. 2021)

Let $u, v \in A^*$. Then, $u \equiv_{\text{baxt}} v$ if and only if u and v share exactly the same content and right and left precedences.

Identities satisfied by baxt

Theorem (Cain, Malheiro, and R. 2021)

For any $n \geq 2$, baxt *and* baxt_n *satisfy exactly the same identities.*

Theorem (Cain, Malheiro, and R. 2021)

The identities $u \approx v$ *satisfied by* baxt *are balanced identities such that, for any* $x \in \mathrm{supp}(u \approx v)$, *the longest prefix of* u *where* x *does not occur has the same content as the longest prefix of* v *where* x *does not occur, and the longest suffix of* u *where* x *does not occur has the same content as the longest suffix of* v *where* x *does not occur.*

Corollary (Cain, Malheiro, and R. 2021)

The decision problem Check-Id(baxt) *belongs to the complexity class* P.

Corollary (Cain, Malheiro, and R. 2021)

The shortest non-trivial identity, with n variables, satisfied by baxt*, is of length* $n+4$.

Example

The identity

$$xyxya_1\dots a_{n-2}yx \approx xyyxa_1\dots a_{n-2}yx$$

is a minimum-length non-trivial identity, with n variables, satisfied by baxt.

Corollary (Cain, Malheiro, and R. 2021)

The variety generated by baxt *is strictly contained in the variety generated by* plac_2.

Finite basis and axiomatic rank of $\mathcal{V}_{\mathsf{baxt}}$

Theorem (Cain, Malheiro, and R. 2021)

$\mathcal{V}_{\mathsf{baxt}}$ *admits a finite basis consisting of the following identities:*

$$xzytxyrxsy \approx xzytyxrxsy; \tag{O}$$

$$xzytxyrysx \approx xzytyxrysx. \tag{E}$$

Proposition (Cain, Malheiro, and R. 2021)

Neither of the identities (O) *or* (E) *is a consequence of the set of non-trivial identities, satisfied by* baxt, *over an alphabet with six variables, excluding itself (but not the other) and equivalent identities.*

Corollary (Cain, Malheiro, and R. 2021)

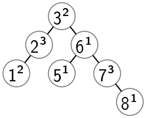
The axiomatic rank of $\mathcal{V}_{\mathsf{baxt}}$ *is* 6*.*

Other plactic-like monoids

Stalactic monoid
stal
Stalactic tableau

1	2	4	3
1	2		3
	2		3

Taiga monoid
taig
Binary search tree with multiplicities



Right patience-sorting monoid
rPS
rPS-tableau

3			7	
2		6	6	
2	4	6	5	
1	3	4	5	7

Cain et al. (2022a) have studied the equational theories and varieties of these monoids by giving faithful representations of these monoids of each finite rank as monoids of upper triangular matrices over certain semirings.

They also showed that neither $\mathcal{V}_{\text{sylv}}$ nor $\mathcal{V}_{\text{baxt}}$ are contained in the join of $\mathfrak{C}$ and any variety generated by a finite monoid.

The stylic monoid styl_n

N-tableau

5	6				
2	5	6			
1	2	3	4	5	6

- Young tableau with rows strictly increasing left to right;
- Each row contained in the one below it;
- Left-justified, longer rows on bottom.

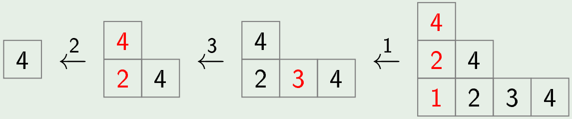
- The right N-algorithm computes a unique N-tableau $\mathrm{P}_{\mathrm{styl}}^{\rightarrow}(u)$ for each word $u \in A^*$.
- Define the congruence $\equiv_{\mathrm{styl}}$ on A^* as follows: For $u, v \in A^*$,

$$u \equiv_{\mathrm{styl}} v \iff \mathrm{P}_{\mathrm{styl}}^{\rightarrow}(u) = \mathrm{P}_{\mathrm{styl}}^{\rightarrow}(v).$$

- The factor monoid $\mathrm{styl}_n := A_n^*/{\equiv_{\mathrm{styl}}}$ is the stylic monoid of finite rank n (Abram and Reutenauer 2022).

Example (Krob–Thibon algorithm)

Computing $\mathrm{P}_{\mathsf{styl}}^{\rightarrow}(4231)$:



- styl_n is a finite quotient of plac_n, defined by the action of words over A_n on the left of columns of Young tableaux, by Schensted left insertion.
- It is presented by the Knuth relations and the relations $a^2 \equiv a$, with $a \in \mathcal{A}_n$.
- It is a finite $\mathcal{J}$-trivial monoid.

Pseudovarieties

- A pseudovariety of monoids is a nonempty class of finite monoids closed under submonoids, homomorphic images and finitary direct products.
- An equational pseudovariety consists of all the finite monoids in some variety.
- An equational pseudovariety is defined by its equational theory.
- A finitely generated pseudovariety is equational.
- $\mathcal{J}_K$ is the pseudovariety in Simon's hierarchy of $\mathcal{J}$-trivial monoids which corresponds to the class of all piecewise testable languages of height k, in Eilenberg's correspondence.
- $\mathcal{J}_K$ is an equational pseudovariety, and its equational theory is the set J_k of all identities $u \approx v$ such that u and v share the same subsequences of length $\leq k$.

- $\mathcal{J}_1$ admits a finite basis, consisting of the following identities:

$$x^2 \approx x \quad \text{and} \quad xy \approx yx.$$

- $\mathcal{J}_2$ admits a finite basis, consisting of the following identities:

$$xyxzx \approx xyzx \quad \text{and} \quad (xy)^2 \approx (yx)^2.$$

- $\mathcal{J}_3$ admits a finite basis, consisting of the following identities:

$$\begin{aligned} xyx^2zx &\approx xyxzx, \\ xyzx^2tz &\approx xyxzx^2tx, \\ zyx^2ztx &\approx zyx^2zxtx, \\ (xy)^3 &\approx (yx)^3. \end{aligned}$$

- $\mathcal{J}_k$ is non-finitely based, for $k \geq 4$.

The tropical semiring and monoids of tropical matrices

- The tropical (max-plus) semiring $\mathbb{T}$ is the set $\mathbb{R} \cup \{-\infty\}$ under the operations $a \oplus b = \max\{a, b\}$ and $a \otimes b = a + b$.
- The set of upper unitriangular $n \times n$ matrices with entries in $\mathbb{T}$ forms a monoid $U_n(\mathbb{T})$ under matrix multiplication induced by the operations in $\mathbb{T}$.
- For each $n \in \mathbb{N}$, the monoid $U_{n+1}(\mathbb{T})$ generates $\mathcal{V}(J_n)$, that is, the variety whose equational theory is J_n (M. Johnson and Fenner 2019).

Tropical representations of the stylic monoids

Define the map $\rho_n \colon A_n^* \to U_{n+1}(\mathbb{T})$ as follows:

$$\rho_n(x)_{i,j} = \begin{cases} 0 & \text{if } i = j; \\ 1 & \text{if } i \leq n+1-x < j; \\ -\infty & \text{otherwise.} \end{cases}$$

for each $x \in A_n$, extending multiplicatively to all of A_n^* and defining $\rho_n(\varepsilon) = I_{(n+1)\times(n+1)}$.

Example

The images of 2 and of 4213 under ρ_4 are, respectively,

$$\begin{bmatrix} 0 & -\infty & -\infty & 1 & 1 \\ -\infty & 0 & -\infty & 1 & 1 \\ -\infty & -\infty & 0 & 1 & 1 \\ -\infty & -\infty & -\infty & 0 & -\infty \\ -\infty & -\infty & -\infty & -\infty & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 & 2 & 2 & 3 \\ -\infty & 0 & 1 & 1 & 2 \\ -\infty & -\infty & 0 & 1 & 2 \\ -\infty & -\infty & -\infty & 0 & 1 \\ -\infty & -\infty & -\infty & -\infty & 0 \end{bmatrix}$$

Proposition (Aird and R. 2022)

The map ρ_n induces a well-defined morphism from styl_n *to* $U_{n+1}(\mathbb{T})$.

Denote by ϱ_n the induced morphism from styl_n to $U_{n+1}(\mathbb{T})$.

Example

The image of $[4213]_{\mathsf{styl}_4}$ under ϱ_4 is the same as that of 4213 under ρ_4, that is,

$$\begin{array}{|c|c|c|c|}\hline 4 \\ \cline{1-1} 2 & 4 \\ \cline{1-2} 1 & 2 & 3 & 4 \\ \hline \end{array} \quad \xmapsto{\varrho_4} \quad \begin{bmatrix} 0 & 1 & 2 & 2 & 3 \\ -\infty & 0 & 1 & 1 & 2 \\ -\infty & -\infty & 0 & 1 & 2 \\ -\infty & -\infty & -\infty & 0 & 1 \\ -\infty & -\infty & -\infty & -\infty & 0 \end{bmatrix}$$

Lemma (Aird and R. 2022)

Let $w \in A_n^*$, $a \in A_n$, *and* $k \in \mathbb{N}$. *Then,* a *occurs in the* k*-th row of* $N(w)$ *if and only if there exists* $j \in \{1, \ldots, n+1\}$, *with* $n+1-a < j$, *such that* $\rho_n(w)_{n+1-a,j} = k$, *and* $\rho_n(w)_{n+2-a,j} = k-1$.

Theorem (Aird and R. 2022)

The morphism $\varrho_n \colon \mathsf{styl}_n \to U_{n+1}(\mathbb{T})$ *is a faithful representation of* styl_n.

Therefore, all identities in J_n are satisfied by styl_n.

Identities satisfied by styl_n

Theorem (Aird and R. 2022)

Let $n \in \mathbb{N}$ and let $u \approx v$ be a non-trivial identity satisfied by styl_n. Then, $u \approx v \in J_n$.

Corollary (Aird and R. 2022)

For each $n \in \mathbb{N}$, styl_n generates the variety $\mathcal{V}(J_n)$ and the pseudovariety $\mathfrak{J}_n$. Furthermore, $\mathcal{V}(\mathrm{styl}_n) \subsetneq \mathcal{V}(\mathrm{styl}_{n+1})$, and styl_n is finitely based if and only if $n \leq 3$.

Corollary (Aird and R. 2022)

For each $n \in \mathbb{N}$, the identity checking problem for styl_n is decidable in linearithmic time. Therefore, Check-Id(styl_n) is in the complexity class P.

Corollary (Aird and R. 2022)

$\mathcal{V}(\mathsf{styl}_n)$ *has uncountably many subvarieties, for $n \in \mathbb{N}$ such that $n \geq 3$.*

Corollary (Aird and R. 2022)

$\mathcal{V}(\mathsf{hypo})$ *is the varietal join $\mathfrak{C} \vee \mathcal{V}(\mathsf{styl}_2)$, and is generated by the free commutative monoid and styl_2.*

Finite basis problem for styl_n with involution

- An involution on a semigroup S is a unary operation $*$ on S such that $(x^*)^* = x$ and $(xy)^* = y^*x^*$.
- An involution semigroup is a semigroup S together with an involution $*$, denoted $(S, *)$.
- The unique order-reversing permutation on a finite ordered alphabet A_n induces an involution $*$ of the stylic monoid of rank n.
- The operation of skew transposition, denoted *, is an involution on the monoid of unitriangular matrices over the tropical semiring.

Proposition (Aird and R. 2022)

The morphism $\varrho_n \colon \mathsf{styl}_n \to U_{n+1}(\mathbb{T})$ *extends to a faithful morphism from* $(\mathsf{styl}_n, *)$ *to* $(U_{n+1}(\mathbb{T}), {}^\star)$.

Proposition (Aird and R. 2022)

For each $n \geq 2$, $(\mathsf{styl}_n, *)$ *satisfies the identity*

$$x^* x^{n-1} \approx x^* x^n,$$

while $(U_{n+1}(\mathbb{T}), {}^\star)$ *does not.*

Theorem (Aird and R. 2022)

The involution monoid $(\mathsf{styl}_n, *)$ *is finitely based if and only if* $n = 1$.

Thank you for your attention!

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