Super plactic structures : combinatorial and rewriting approaches

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Séminaire réécriture algébrique

Outline

I. Introduction and motivation

II. Super jeu de taquin

III. Super-RSK correspondence with symmetry

IV. Super Littlewood–Richardson rule

V. Coherence by insertion and perspectives

I. Introduction and motivation

The Littlewood–Richardson rule

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$T =$

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2	3	5	
4	4		
6			

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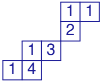

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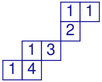

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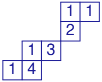

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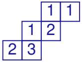

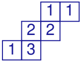

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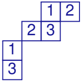

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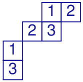

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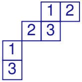

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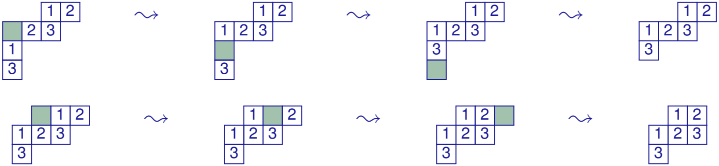

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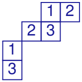

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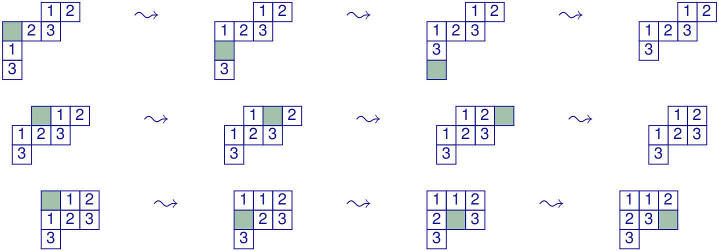

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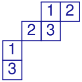

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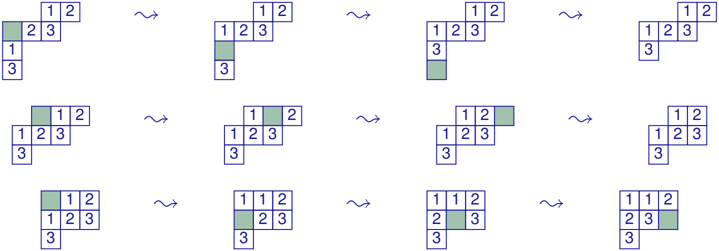

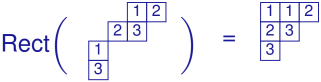

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- **Knuth relations**, (Knuth, '70) : $\sim_K$ the congruence on $[n] := \{1, \ldots, n\}$:

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- **Symmetry property**, (Viennot, '77, Fulton, '97) :

$$w^{\text{inv}} \overset{RSK}{\longleftrightarrow} (Q(w), T(w))$$

Super plactic structures

Super plactic structures

- **Signed alphabet** $(\mathcal{S}, ||.||)$:
  - $\mathcal{S}$ a finite or countable totally ordered set,
  - $||.|| : \mathcal{S} \longrightarrow \mathbb{Z}_2 = \{0, 1\}$ be any map,
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- **Super plactic monoid** $\mathbf{P}(\mathcal{S})$ over $\mathcal{S}$, (La Scala, Nardozza, Senato, '06) :

$$
\begin{aligned}
xzy &= zxy, \quad \text{with } x = y \text{ only if } ||y|| = 0 \text{ and } y = z \text{ only if } ||y|| = 1,\\
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  for any $x \leq y \leq z$ in $\mathcal{S}$.
- When $\mathcal{S}_0 < \mathcal{S}_1$, (Benkart, Kang, Kashiwara, '00) :

$$\{\overline{m} < \ldots < \overline{1} < 1 < \ldots < n\}$$

  - The congruence $\sim_{\mathbf{P}(\mathcal{S})}$ describes **crystal isomorphism** of the crystal graph for $\mathfrak{gl}_{m,n}$.

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II. Super jeu de taquin

Super jeu de taquin

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- $\mathrm{Yt}(\mathcal{S})$ set of **super tableaux** : $\mathcal{S}_0 = \{\text{even integers}\}$, $\mathcal{S}_1 = \{\text{odd integers}\}$

t =

1	2	2	3	4	4	4
1	3	5	7	8	8	
1	3	5	7			
4	4					

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$$R_{col}(t) = 4111\ 4332\ 552\ 773\ 84\ 84\ 4$$
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$$t = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 2 & 2 & 3 & 4 & 4 & 4 \\ \hline 1 & 3 & 5 & 7 & 8 & 8 \\ \cline{1-6} 1 & 3 & 5 & 7 \\ \cline{1-4} 4 & 4 \\ \cline{1-2}\end{array}$$

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Proposition.(H., '22) : $R_{col}(t) \sim_{\mathbf{P}(\mathcal{S})} R_{row}(t)$

Super jeu de taquin

- $\mathrm{Yt}(\mathcal{S})$ set of **super tableaux** : $\mathcal{S}_0 = \{\text{even integers}\}$, $\mathcal{S}_1 = \{\text{odd integers}\}$

t =

1	2	2	3	4	4	4
1	3	5	7	8	8	
1	3	5	7			
4	4					

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- **Insertion on super tableaux**, (La Scala, Nardozza, Senato, '06) :

1	2	2	3	4	4	4
1	3	5	7	8	8	
1	3	5	7			
4	4					

$\leftsquigarrow 2 =$

1	2	2	2	4	4	4
1	3	5	7	8	8	
1	3	5	7			
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4						

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$$2 \rightsquigarrow \begin{array}{|c|c|c|c|c|c|c|}\hline 1&2&2&3&4&4&4\\\hline 1&3&5&7&8&8\\\cline{1-6} 1&3&5&7\\\cline{1-4} 4&4\\\cline{1-2}\end{array} = \begin{array}{|c|c|c|c|c|c|c|}\hline 1&2&2&3&4&4&4\\\hline 1&3&4&5&7&8&8\\\hline 1&3&5&7\\\cline{1-4} 2&4\\\cline{1-2}\end{array}$$

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- **Super tableau constructor** : $w = x_1 \ldots x_k$

$$T(w) := (\emptyset \leftsquigarrow w) = ((\ldots(\emptyset \leftsquigarrow x_1) \leftsquigarrow \ldots) \leftsquigarrow x_k).$$

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4						

$2 \rightsquigarrow$

1	2	2	3	4	4	4
1	3	5	7	8	8	
1	3	5	7			
4	4					

$=$

1	2	2	3	4	4	4
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4						

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1	2	2	3	4	4	4
1	3	5	7	8	8	
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$=$

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1	3	5	7			
2	4					

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Super jeu de taquin

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- The **cross-section** property, (La Scala, Nardozza, Senato, '06) :

$$w \sim_{\mathbf{P}(\mathcal{S})} v \quad \text{if and only if} \quad T(w) = T(v)$$

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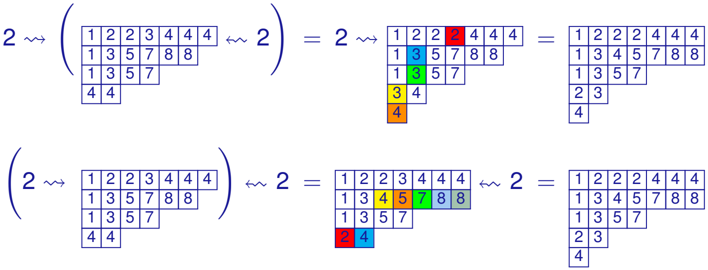
- The insertion $\leftsquigarrow$ and $\rightsquigarrow$ **commute** :



Super jeu de taquin

Super jeu de taquin

- **Super skew tableaux** : $\mathcal{S}_0 = \{\text{odd integers}\}$, $\mathcal{S}_1 = \{\text{even integers}\}$

		1	1	2
	3	3		
4	5			
4				

- **shape** : $(5, 3, 2, 1)/(2, 1)$

Super jeu de taquin

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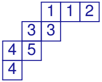

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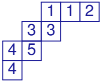

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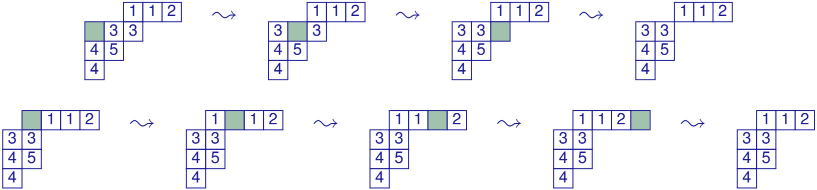

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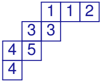

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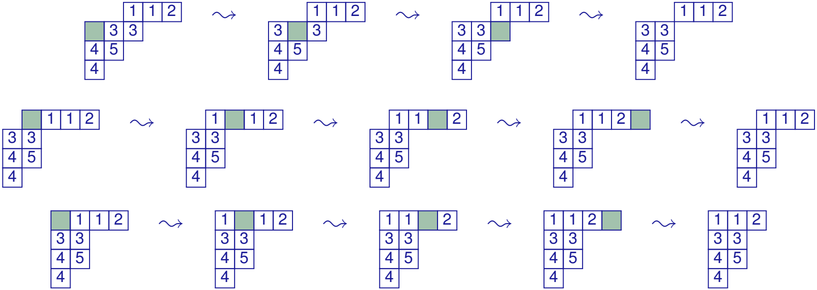

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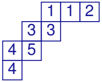

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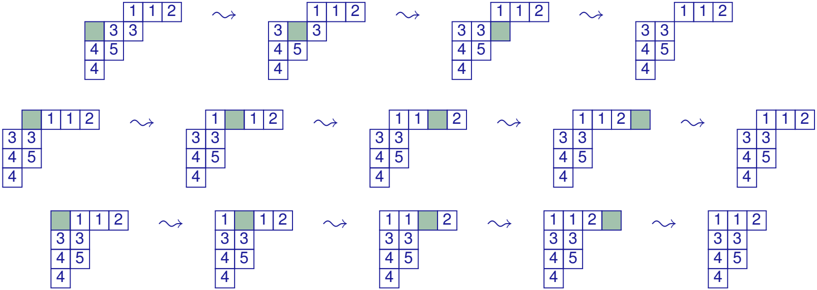

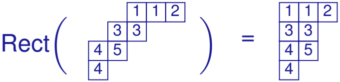

Super jeu de taquin

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Theorem (H. '22) :

- $\mathrm{Rect}(S)$ is the **unique** super tableau such that

$$R_{col}(\mathrm{Rect}(S)) \sim_{\mathbf{P}(\mathcal{S})} R_{col}(S).$$

- **Confluence of super jeu de taquin** : Rect does not depend on the order in which we choose inner corners.

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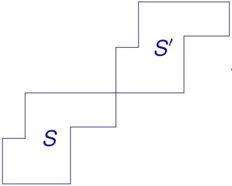

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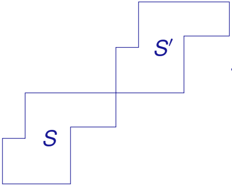

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Proposition (H. '22) : $\text{Rect}([S, S']) = S \star_{\text{Yt}(\mathcal{S})} S'$.

III. Super-RSK correspondence with symmetry

Super-RSK correspondence with symmetry

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- **Signed two rowed array**.

  $\mathcal{S}_0 = \mathcal{S}_1' = \{\text{even numbers}\}$, $\mathcal{S}_1 = \mathcal{S}_0' = \{\text{odd numbers}\}$:

$$w = \begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 \\ 3 & 2 & 1 & 2 & 4 & 3 & 1 & 2 \end{pmatrix}$$

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- **Super-RSK correspondence** :

$$(\emptyset\,,\ \emptyset),\quad \left(\begin{array}{|c|}\hline 3\\\hline\end{array}\,,\ \begin{array}{|c|}\hline 1\\\hline\end{array}\right),\quad \left(\begin{array}{l} 2\\ 3\end{array}\,,\ \begin{array}{l} 1\\ 1\end{array}\right),\quad \left(\begin{array}{l} 1\\ 2\\ 3\end{array}\,,\ \begin{array}{l} 1\\ 1\\ 2\end{array}\right),\quad \left(\begin{array}{ll} 1 & 2\\ 2\\ 3\end{array}\,,\ \begin{array}{ll} 1 & 2\\ 1\\ 2\end{array}\right),$$

$$\left(\begin{array}{lll} 1 & 2 & 4\\ 2\\ 3\end{array}\,,\ \begin{array}{lll} 1 & 2 & 3\\ 1\\ 2\end{array}\right),\quad \left(\begin{array}{lll} 1 & 2 & 3\\ 2 & 4\\ 3\end{array}\,,\ \begin{array}{lll} 1 & 2 & 3\\ 1 & 3\\ 2\end{array}\right),\quad \left(\begin{array}{lll} 1 & 1 & 3\\ 2 & 4\\ 2\\ 3\end{array}\,,\ \begin{array}{lll} 1 & 2 & 3\\ 1 & 3\\ 2\\ 4\end{array}\right),$$

$$\left(T(w) = \begin{array}{lll} 1 & 1 & 2\\ 2 & 3\\ 2 & 4\\ 3\end{array}\,,\ Q(w) = \begin{array}{lll} 1 & 2 & 3\\ 1 & 3\\ 2 & 4\\ 4\end{array}\right).$$

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Theorem (H. '22). There is a one-to-one correspondence :

$$w \overset{\text{RSK}}{\longleftrightarrow} (T(w), Q(w))$$

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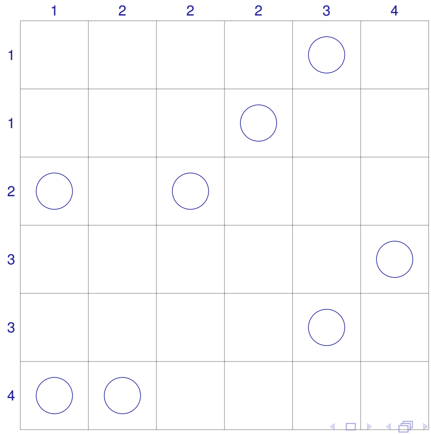

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	1	2	2	2	3	4
1					1	
1				1		
2	1		2			
3						3
3					3	
4	2	3				

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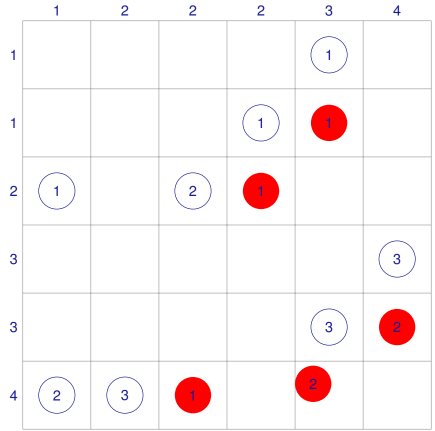

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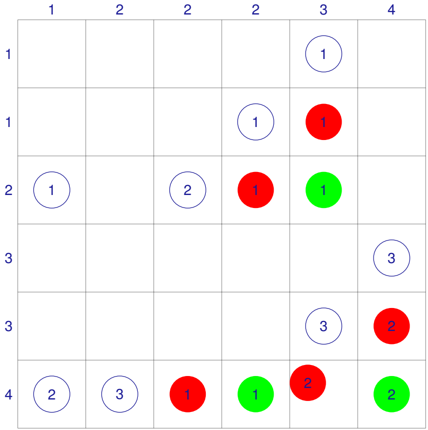

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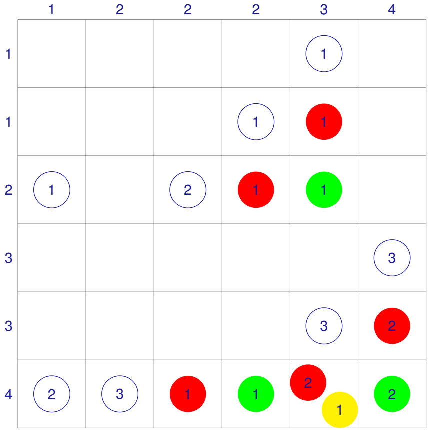
$$w = \begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 \\ 3 & 2 & 1 & 2 & 4 & 3 & 1 & 2 \end{pmatrix}$$

- **Matrix-ball construction :**



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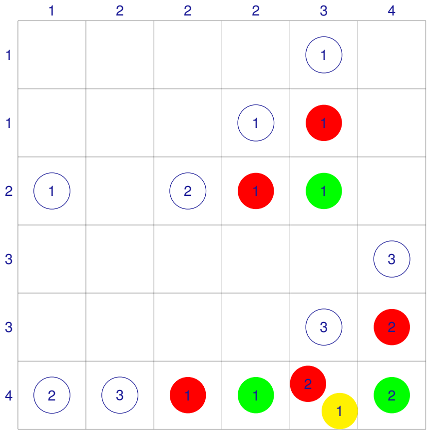

$$\left(T(w) = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 3 & \\ \cline{1-2} 2 & 4 & \\ \cline{1-2} 3 & & \\ \cline{1-1} \end{array}, \quad Q(w) = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & 3 & \\ \cline{1-2} 2 & 4 & \\ \cline{1-2} 4 & & \\ \cline{1-1} \end{array} \right)$$

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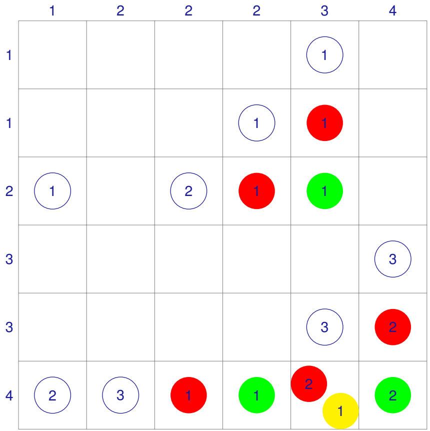

Super-RSK correspondence with symmetry

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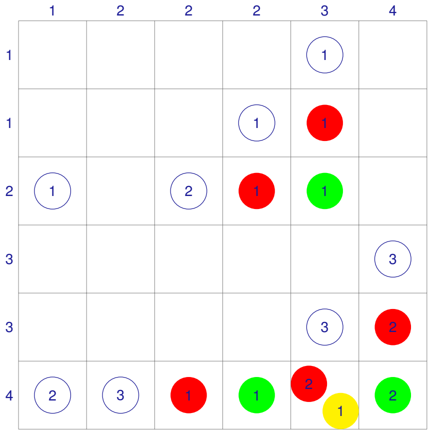

$$w^{\text{inv}} = \begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 \\ 2 & 4 & 4 & 2 & 1 & 1 & 3 & 3 \end{pmatrix}$$

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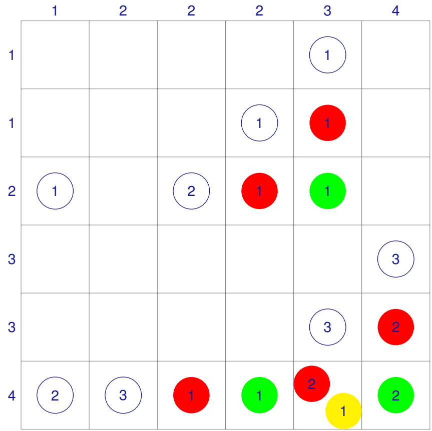

$$w^{\text{inv}} = \begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 \\ 2 & 4 & 4 & 2 & 1 & 1 & 3 & 3 \end{pmatrix}$$

$$\updownarrow$$

$$\left(Q(w) = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & 3 \\ \cline{1-2} 2 & 4 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}, \quad T(w) = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 3 \\ \cline{1-2} 2 & 4 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array} \right)$$

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$$w \overset{RSK}{\longleftrightarrow} (T(w), Q(w)) \iff w^{\text{inv}} \overset{RSK}{\longleftrightarrow} (Q(w), T(w)).$$

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$$(\emptyset, \emptyset), \quad \left(\begin{array}{|c|}\hline 2\\\hline\end{array}, \begin{array}{|c|}\hline 4\\\hline\end{array}\right), \quad \left(\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}, \begin{array}{|c|c|}\hline 4&4\\\hline\end{array}\right), \quad \left(\begin{array}{cc} 1&2\\ 3& \end{array}, \begin{array}{cc} 3&4\\ 4& \end{array}\right), \quad \left(\begin{array}{cc} 1&2\\ 3& \\ 4& \end{array}, \begin{array}{cc} 3&4\\ 3& \\ 4& \end{array}\right),$$

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$$\left(T(w) = \begin{array}{ccc} 1&1&2\\ 2&3& \\ 2&4& \\ 3&& \end{array}, \quad Q(w) = \begin{array}{ccc} 1&2&3\\ 1&3& \\ 2&4& \\ 4&& \end{array}\right).$$

IV. Super Littlewood–Richardson rule

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- **Super Littlewood–Richardson coefficient** $c^{\nu}_{\lambda,\mu}$: number of ways a given super tableau t of shape ν can be written as

$$t = t' \star_{\mathrm{Yt}(\mathcal{S})} t''$$

where t' is of shape λ and t'' is of shape μ.

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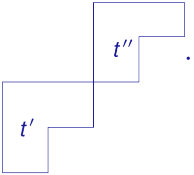

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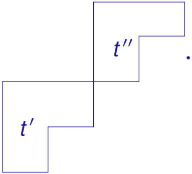

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Theorem (H. '22).

- $c^{\nu}_{\lambda,\mu}$ is equal to the number of super skew tableaux of shape ν/λ whose rectification is a given tableau of shape μ.
- $c^{\nu}_{\lambda,\mu}$ does not depend on t, and depends only on λ, μ and ν.

Super Littlewood–Richardson rule

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- T super tableau $\rightsquigarrow x^T = \prod_{i=1}^{m} x_i^{\text{number of times } i \text{ occurs in } T}$

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & 6 \\ \hline 1 & 3 & 5 & \\ \cline{1-3} 2 & 3 & & \\ \cline{1-2} 4 & & & \\ \cline{1-1} \end{array}$$

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<table>
<tr><td>1</td><td>2</td><td>2</td><td>6</td></tr>
<tr><td>1</td><td>3</td><td>5</td><td></td></tr>
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<tr><td>4</td><td></td><td></td><td></td></tr>
</table>

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Theorem (H. '22). The following equalities hold in $R_\mathcal{S}$:

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda,\mu} s_\nu \qquad s_{\nu/\lambda} = \sum_\mu c^\nu_{\lambda,\mu} s_\mu .$$

V. Coherence by insertion and perspectives

Coherence by insertion and perspectives

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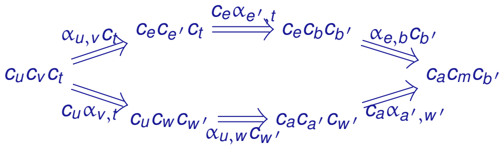
- Super column coherent presentation of the super plactic monoid, (H., ’22) :
  - generators : super columns
  - rules : $\alpha_{u,v} : c_u c_v \Rightarrow c_w c_{w'}$
  - homotopy generators :



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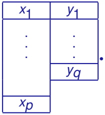

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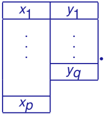

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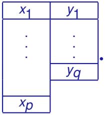

- Denote by $u \overset{\times}{\;} v$ the other cases :
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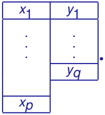

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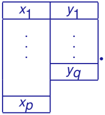

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Lemma (H., 22). Let u and v two super columns such that $u \overset{\times}{} v$. The super tableau $T(uv)$ consists of at most two super columns.

Coherence by insertion and perspectives

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Theorem (H., 22). The 2-polygraph $\mathrm{Col}_2(\mathcal{S})$:

- 1-cells : super column c_u,
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is a convergent presentation of the super plactic $P(\mathcal{S})$.

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$$c_u c_v c_t \xRightarrow{\alpha_{u,v} c_t} c_e c_{e'} c_t \xRightarrow{c_e \alpha_{e',t}} c_e c_b c_{b'} \xRightarrow{\alpha_{e,b} c_{b'}} c_a c_m c_{b'}$$

$$c_u c_v c_t \xRightarrow{c_u \alpha_{v,t}} c_u c_w c_{w'} \xRightarrow{\alpha_{u,w} c_{w'}} c_a c_{a'} c_{w'} \xRightarrow{c_a \alpha_{a',w'}} c_a c_m c_{b'}$$

such that $u \,{}^{\times}\, v \,{}^{\times}\, t$, is a coherent presentation of $P(\mathcal{S})$.

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Homotopical reduction : remove the redundant generators by taking into account the relations between the relations of the elements of the homotopy basis in higher dimension.

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$$c_x c_v c_t \overset{c_x \alpha_{v,t}}{\Longrightarrow} c_x c_w c_{w'} \underset{\alpha_{x,w} c_{w'}}{\Longrightarrow} c_a c_{a'} c_{w'} \overset{c_a \alpha_{a',w'}}{\Longrightarrow} c_a c_d c_{b'}$$

for every letter x in $\mathcal{S}$ and super columns v and t such that $x \overset{\times}{\;} v \overset{\times}{\;} t$, is a coherent presentation of $P(\mathcal{S})$.

Coherence by insertion and perspectives

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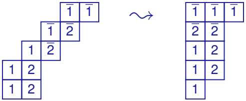
- **Super Littlewood–Richardson skew tableaux** using genuine highest weight super tableaux



  - a new combinatorial version of the super Littlewood-Richardson rule,
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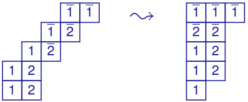

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- The **left** and **right keys** on super tableaux :

  - The right and left key of a semistandard Young tableau were introduced by Lascoux and Schützenberger in 1990.
  - The right key is a tool used to find Demazure characters for the general Lie algebra using the jeu de taquin.
  - **Demazure atoms** and **characters** for the general Lie superalgebra !

Coherence by insertion and perspectives

- **A super-RSK correspondence with symmetry and the super Littlewood–Richardson rule for super tableaux**, in preparation, 2022.
- **Coherent presentations of super plactic monoids of type A by insertions**, arxiv :2112.09633, submitted, 2021.
- **Super jeu de taquin and combinatorics of super tableaux of type A**, International Journal of Algebra and Computation, arxiv :2105.07819, 2022.

Thank you for your attention !