Coherence by decreasingness for monoids

Séminaire de réécriture

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Plan

I. Motivation

II. Coherence by decreasingness for ARS

III. Coherence by decreasingness for SRS

IV. Coherence by decreasingness for $\mathcal{K}$-SRS

V. Perspectives

I. Motivation

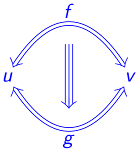
Computing homotopy bases for finitely presented monoids

Let M be a monoid

- generated by a finite set X of generators,
- submitted to a finite set $R \subseteq X^* \times X^*$ of oriented relations, called **rewriting rules**.

Question: What is an homotopy basis?

- A **syzygy**: a relation between relations



- An **homotopy basis**: a family of syzygies that generates all syzygies of (X, R).

Convergent presentations of monoids

Let M be a monoid finitely presented.

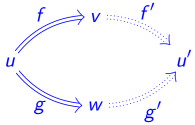
Question: How to compute an homotopy basis?

Convergent presentation (X, R) of M:

- **Terminating**: all computations end.
  - In general define: left degree-wise lexicographic order. Fix an order $\prec$ on elements of X.

$$
\begin{aligned}
u >_{\text{lex}} v \text{ iff } \ell(u) > \ell(v) \text{ or } \ell(u) = \ell(v) \\
\text{and } u = x_1 x_2 \ldots x_{k-1} x_k x_{k+1} \ldots x_n, \\
v = x_1 x_2 \ldots x_{k-1} y_k y_{k+1} \ldots y_n \text{ with } y_k \prec x_k.
\end{aligned}
$$

- **Confluent**: the computation converges to the same result (if it exists).

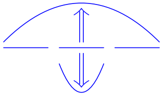


Squier's completion theorem, '94

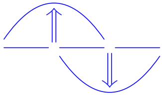
Let $M = \langle X, R \rangle$ be a monoid presented by a finite convergent presentation.

- A **critical branching**: an overlapping of two rewriting rules that is minimal with respect to the rewriting context.

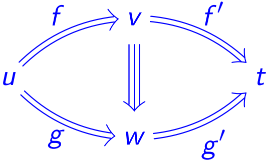
  There are two shapes of critical branchings:

  and

- The family of syzygies formed by **generating confluences**

  indexed by critical branchings, form an homotopy basis.

Another method to compute homotopy bases

Let $M = \langle X, R \rangle$ be a monoid finitely presented such that

- each word M is represented by a **recurrent form**, i.e. a normal form modulo cycles.
- each critical pair is **decreasing**, which generalizes the confluence property by adding a well-founded labelling on rewriting steps, [van Oostrom, '94].

Question: How to compute an homotopy basis of (X, R)?

Example 1: Braid monoid B_3^+

Presentation of B_3^+

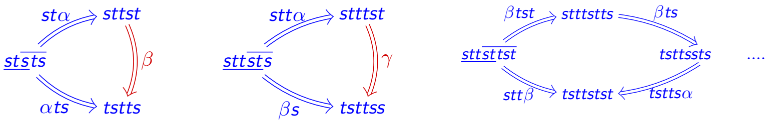
$$\langle s, t \mid \alpha : sts \Rightarrow tst \rangle.$$

- Termination: degree lexicographic order on $s > t$.
- One non confluent critical branching:

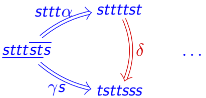
$$\underline{sts}\overline{ts} \;\overset{st\alpha}{\Rightarrow}\; sttst \qquad \underline{sts}\overline{ts} \;\overset{\alpha ts}{\Rightarrow}\; tstts$$

Example 1: Braid monoid B_3^+

Knuth-Bendix completion: it gives, by adding



a convergent presentation of B_3^+ on the two generators s and t, which is infinite



Theorem. *B_3^+ does not admit a finite convergent presentation with the two generators s and t, Kapur & Narendran, '85.*

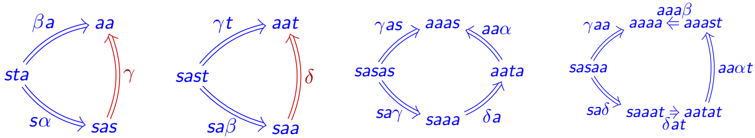
Example 1: Braid monoid B_3^+

Presentation of B_3^+ by adjunction of a new generator a

$$\langle s, t, a \mid \alpha : ta \Rightarrow as, \beta : st \Rightarrow a \rangle$$

standing for the product st.

- Termination: degree lexicographic order on $s > t > a$.
- Knuth-Bendix completion:



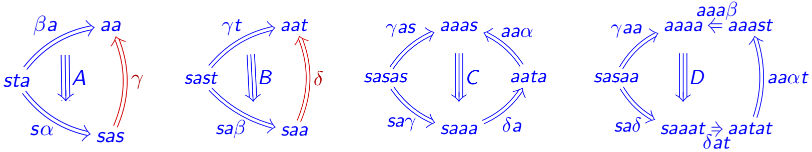
The SRS $< s, t, a \mid ta \overset{\alpha}{\Rightarrow} as \;,\; st \overset{\beta}{\Rightarrow} a \;,\; sas \overset{\gamma}{\Rightarrow} aa \;,\; saa \overset{\delta}{\Rightarrow} aat \;>$ is a convergent presentation of B_3^+.

Example 1: Braid monoid B_3^+

By **Squier's completion** of

$$\langle s, t, a \mid \alpha : ta \Rightarrow as, \beta : st \Rightarrow a, \overset{\gamma}{\Rightarrow} aa \ , \ saa \overset{\delta}{\Rightarrow} aat \rangle,$$

we obtain an homotopy basis formed by these generating confluences



Another method to compute an homotopy basis for B_3^+

We consider a presentation of B_3^+, [Alleaume-Malbos, '17]:

$$\Sigma(\mathrm{B}_3^+) = \langle s, t \mid \alpha : sts \Rightarrow tst,\ \beta : tst \Rightarrow sts \rangle.$$

by working on the orientation of the rewriting rules instead of adding new generators.

Question:
Can we compute an homotopy basis of $\Sigma(\mathrm{B}_3^+)$?

Example 2: Plactic monoid P_n

Plactic monoid P_n

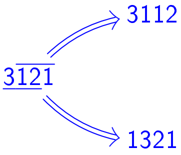
- generated by $1 < 2 < \dots < n$
- and submitted to Knuths relations

$$x \leq y < z, \qquad yzx \Rightarrow yxz$$

$$x < y \leq z, \qquad zxy \Rightarrow xzy$$

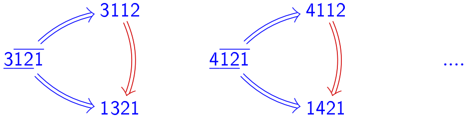
with $x, y, z \in \{1, 2, 3, \dots, n\}$.

- Termination: degree lexicographic order on $1 < 2 < \dots < n$.
- Some non confluent critical branchings:



Example 2: Plactic monoid P_n

Knuth-Bendix completion: it gives, by adding



a convergent presentation of P_n on the generators $1, \ldots, n$, which is infinite.

> ***Theorem.*** *$\forall n > 3$, P_n does not admit a finite convergent presentation by Knuth-Bendix completion of the Knuth presentation with the degree lexicographic order, Kubat and Okninski, '14.*

Example 2: plactic monoids P_n

- **Young tableau** over $\{1, \ldots, n\}$:

2	2	3	5	5
3	3	4	6	
4	5			

  $\mathrm{Col}(n)$: set of columns.

- **Schensted algorithm:**

  any word w in $\{1, \ldots, n\}^* \rightsquigarrow$ Young tableau $P(w)$.

- Notation: any two columns $u^{\times}v$ iff $T = c_u c_v$ is not a Young tableau.

- Oriented relations on $Col(n)^*$:

$$\mathrm{Col}_2(n) \; = \left\{ \; c_u c_v \Rightarrow c_w c_{w'} \; \middle| \; u^{\times} v \in Col(n) \text{ and } P(uv) = c_w c_{w'} \right\}$$

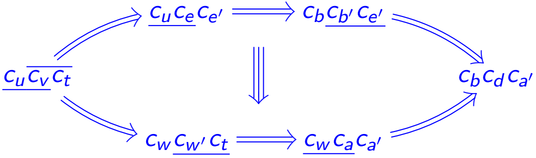
Example 2: Plactic monoid P_n

- The **column presentation** $Col_2(n)$ is a convergent presentation of P_n.

- By **Squier's completion** of

$$\mathrm{Col}_2(n) \;=\{\; c_u c_v \Rightarrow c_w c_{w'} \;\big|\; u^{\times} v \in Col(n) \text{ and } P(uv) = c_w c_{w'} \big\},$$

  we obtain an homotopy basis, denoted $Col_3(n)$, formed by these confluences



  indexed by critical branchings, [Hage-Malbos, '16].

Another method to compute an homotopy basis for P_n

We consider a presentation of P_n:

$$\Sigma(\mathrm{P}_n) = \langle 1, \ldots, n \mid yzx \overset{\alpha_{x,y,z}}{\Rightarrow} yxz, x \leq y < z; \;\; xzy \overset{\beta_{x,y,z}}{\Rightarrow} zxy, x < y \leq z \rangle.$$

by working on the orientation of the rewriting rules instead of adding new generators.

Questions:
Can we compute an homotopy basis of $\Sigma(\mathrm{P}_n)$? Is it smaller than the homotopy basis $Col_3(n)$ of the column presentation $Col_2(n)$?

II. Coherence by decreasingness for ARS

One-dimensional polygraphs

A **1-polygraph** X: modelization of abstract rewriting system.

- set of 0-generators X_0,
- set of 1-generators X_1, **called rewriting steps**,
- source and target maps

$$X_0 \underset{t_0}{\overset{s_0}{\Longleftarrow}} X_1 .$$

Free categories over a 1-polygraph X:

- **free 1-category** X_1^* whose morphisms:

$$x_0 \overset{u_1}{\to} x_1 \overset{u_2}{\to} x_2 \overset{u_3}{\to} \dots \overset{u_n}{\to} x_n$$

  with $u_i \in X_1$.

- **free (1, 0)-category** $X_1^\top$ whose morphisms:

$$x_0 \overset{u_1}{\longleftrightarrow} x_1 \overset{u_2}{\longleftrightarrow} x_2 \overset{u_3}{\longleftrightarrow} \dots \overset{u_n}{\longleftrightarrow} x_n$$

  with $u_i : x_{i-1} \to x_i$ or $u_i : x_i \to x_{i-1}$.

van Oostrom's decreasingness theory: well-founded labelling

A **Well-founded labelled 1-polygraph** is a data $(X, I, \leq_I, \psi)$ made of:

- a 1-polygraph X,
- well-founded ordered set $(I, \leq_I)$ of labels,
- **well-founded labelling**:

$$\begin{array}{rccc} \psi: & X_1 & \longmapsto & (I, \leq_I) \\ & u & \longmapsto & \psi(u). \end{array}$$

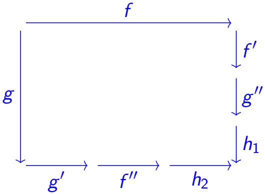
Given a 1-cell $f = x_0 \overset{u_1}{\to} x_1 \overset{u_2}{\to} x_2 \overset{u_3}{\to} \ldots \overset{u_n}{\to} x_n \quad \in X_1^*$, we denote by

$$L^I(f) = \{\psi(u_1), \ldots, \psi(u_n)\}$$

the set of labels of rewriting steps in f.

Locally decreasingness

- A **local branching** (f, g): a pair of rewriting steps f and g with $s_0(f) = s_0(g)$.
- A **decreasing confluence diagram** of (f, g) is defined by



with $f', g', f'', g'', h_1, h_2 \in X_1^*$ and such that

  - $k < \psi(f)$, for all k in $L^l(f')$,
  - g'' is an identity or a 1-generator labelled by $\psi(g') = \psi(g)$,
  - $k < \psi(f)$ or $k < \psi(g)$, for all k in $L^l(h_1)$.
  - symmetrically for g', f'' and h_2.

van Oostrom's confluence theorem

Let $(X, I, \leq_I, \psi)$ be a well-founded labelled 1-polygraph.

- A **(finite) multiset** over I: a function $A : I \to \mathbb{N}$ such that the set $\{i \in I \mid A(i) \neq 0\}$ is finite.
  - We denote $M(I)$ the set of finite multisets over I.
  - Rq: we generalize $\leq_I$ to a well-founded order $\leq_{mul}$ on $M(I)$.
- **lexicographic maximum measure**: the multiset of the lexicographically maximal step labels [van Oostrom, '94].

$$\begin{array}{rccc} |.| : & X_1^* & \longmapsto & M(I) \\ & f & \longmapsto & |f| \end{array}$$

- Theorem: Any locally decreasing well-founded polygraph is confluent, [van Oostrom, '94].

Recurrent forms, [Zilli, '84]

Let X be a 1-polygraph.

- The X-**congruence**: equivalence relation on X_0 defined by

$$x \approx^X y \quad \Longleftrightarrow \quad \exists\, f : x \to y \in X_1^\top.$$

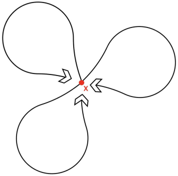
  We denote
  - Equivalence set: $\overline{X} = X_0 / \approx^X$.
  - Canonical projection: $\pi : X_0 \to \overline{X}$.

- A **recurrent 0-generator** x of a 1-polygraph X: for each 1-cell

$$f : x \to y \in X_1^*,$$

  there exists a 1-cell

$$g : y \to x \in X_1^*.$$



Recurrence property

- A **section** of a 1-polygraph X: a section of π

$$r: \begin{array}{ccc} \overline{X} & \longmapsto & X_0 \\ \overline{x} & \longmapsto & r(\overline{x}), \end{array} \qquad \pi(r(\overline{x})) = \overline{x}.$$

  - We denote $r(\overline{x}) = \widetilde{x}$.
  - a **recurrent section** r: for each $x \in X_0$, $\widetilde{x}$ is recurrent.

- A **recurrent 1-polygraph** X: admits a recurrent section
  - each equivalence class $\overline{x}$ is represented by a recurrent 0-generator $\widetilde{x}$.

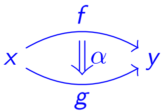
coherent extensions of 1-polygraphs

- A ***sphere*** in $X_1^\top$: a pair (f,g) of 1-*cells* such that

$$s_0(f) = s_0(g) \qquad \text{and} \qquad t_0(f) = t_0(g).$$

  - Denote by $\mathrm{Sph}(X_1^\top)$ the set of spheres of $X_1^\top$.

- A **cellular extension** of $X_1^\top$: a set X_2 with a map $X_2 \to \mathrm{Sph}(X_1^\top)$.
  - Elements of X_2 are called 2-**generators**



- A $(2,0)$-**polygraph** X is defined by
  - a 1-polygraph (X_0, X_1),
  - a cellular extension X_2 of $X_1^\top$

coherent extensions of 1-polygraphs

Let X be a $(2,0)$-polygraph.

- **Free $(2,0)$-category $X_2^\top$** on X.
  - its 2-cells are generated via $\star_0$ and $\star_1$ of 2-generators in X_2, of their inverses and of identities of 1-cells of $X_1^\top$.

- An **homotopy basis** X_2: a cellular extension of $X_2^\top$ such that for any sphere (f,g), there exists a 2-cell

$$A : f \Rightarrow g \in X_2^\top$$

- A **coherent $(2,0)$-polygraph** X is defined by
  - a 1-polygraph (X_0, X_1),
  - an homotopy basis X_2.

Coherence properties

A **coherently recurrent** $(2,0)$-**polygraph** is defined by

- a recurrent 1-polygraph (X_0, X_1),
- a recurrent section $r(\overline{x}) = \widetilde{x}$ of (X_0, X_1),
- a **recurrent cellular extension** X_2 of $X_1^\top$: for any 1-cell

$$f : \widetilde{x} \to \widetilde{x} \in X_1^*,$$

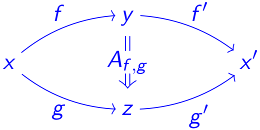
there exists a 2-cell

$$A : f \Rightarrow 1_{\widetilde{x}} \in X_2^\top.$$

Coherence properties

A **coherently locally decreasing** $(2,0)$-**polygraph**:

- a well-founded labelled 1-polygraph (X_0, X_1),
- a well-founded labellinf ψ of (X_0, X_1),
- a **decreasing cellular extension** X_2 of $X_1^\top$: for any local branching (f, g), there exists a decreasing confluence diagram



and a 2-cell $A_{f,g}$ in $X_2^\top$.

Main theorem

Theorem. *Let* X *be a coherently recurrent* $(2, 0)$*-polygraph.*
If X *is locally coherently decreasing, then* X *is coherent.*

III. Coherence by decreasingness for SRS

Application: braid monoid B_3^+

- **Presentation** of B_3^+

$$\Sigma(B_3^+) = \langle s, t \mid \alpha : sts \Rightarrow tst,\ \beta : tst \Rightarrow sts \rangle.$$

- **Rule-labelling** of $\Sigma(B_3^+)$, [van Oostrom, '08].
  - well-founded labelling of **the set of rewriting steps** Σ_{step}

$$\psi : \Sigma_{step} \longmapsto (\mathbb{N}, \leq),$$

  - invariance with the **algebraic context**

$$\psi(u\alpha v) = \psi(\alpha) \quad \text{and} \quad \psi(u\beta v) = \psi(\beta), \quad u, v \in \{s, t\}^*,$$

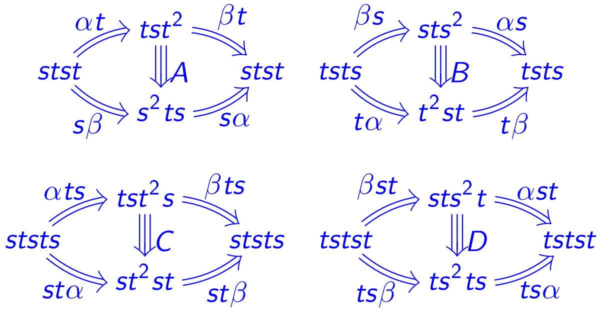
- **constant rule-labelling**

$$\psi(u\alpha v) = \psi(\alpha) = 1 \quad \text{and} \quad \psi(u\beta v) = \psi(\beta) = 1, \quad u, v \in \{s, t\}^*.$$

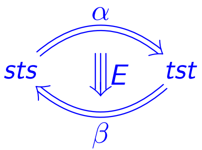
Application: braid monoid B_3^+

Extension of $\Sigma(B_3^+)$ by

- **generating decreasing confluence diagrams**



- a **generating cycle**



Application: braid monoid B_3^+

- The **extended presentation** of B_3^+

$$\langle s, t \mid \alpha : sts \Rightarrow tst, \beta : tst \Rightarrow sts \mid A, B, C, D, E \rangle$$

  is **coherent**.

- By **homotopical reduction**:

$$\langle s, t \mid \alpha, \cancel{\beta} \mid \cancel{A}, \cancel{B}, \cancel{C}, \cancel{D}, \cancel{E} \rangle.$$

  with $A > B > C > D > E$ and $\beta > \alpha$.

- A coherent presentation of B_3^+

$$\langle s, t \mid \alpha : sts \Rightarrow tst \mid \emptyset \rangle.$$

IV. Coherence by decreasingness for $\mathcal{K}$-SRS

Application: plactic monoid P_n

$\mathcal{K}$-**Presentation** $\Sigma(\mathsf{P}_n)$ of P_n

- a set $[n] = \{1, 2, \ldots, n\}$ of generators,

- a **crystal basis** on $[n]$

$$1 \xrightarrow{1} 2 \xrightarrow{2} \ldots \xrightarrow{n-2} n-1 \xrightarrow{n-1} n,$$

- a set of oriented Knuth's relations

$$x \leq y < z, \qquad \alpha_{x,y,z} : yzx \Rightarrow yxz$$

$$x < y \leq z, \qquad \beta_{x,y,z} : xzy \Rightarrow zxy$$

with $x, y, z \in [n]$.

Application: compatibility with Kashiwara's operators

Kashiwara's operators on $[n]$

- For each i in $\{1, 2, \ldots, n-1\}$

$$f_i(i) = i + 1 \qquad \text{and} \qquad e_i(i+1) = i$$

Kashiwara's operators on $[n]^*$

- Example: compute $f_1(121)$

$$\begin{array}{ccc} 1 & 2 & 1 \\ + & - & + \\ & & + \end{array}$$

Then, $f_1(121) = 12f_1(1) = 122$.

Highest weight word w in $[n]^*$: $e_i(w)$ not defined for each $i \in \{1, \ldots n-1\}$.

Application: compatibility with Kashiwara's operators

- Remark: the image of a rewriting rule by Kashiwara's operators is a rewriting rule in the same direction.

  Example: Let $\alpha_{1,1,2} : 121 \Rightarrow 112$. We have:

  - $f_1(121) = 122$ and $f_1(112) = 212$,
  - $\beta_{1,2,2} : 122 \Rightarrow 212$,

  Then,
  $$f_1(\alpha_{1,1,2}) = \beta_{1,2,2} : f_1(121) \Rightarrow f_1(112).$$

- We obtain a $\mathcal{K}$-2-**polygraph**, [Uran, '22], presenting P_n.

- Remark: The Young tableaux are recurrent forms.

Application: Well-fouded labelling of $\Sigma(\mathrm{P}_n)$

A **well-founded labelling** on the set of rewriting steps Σ_{step}

$$\psi : \Sigma_{step} \longmapsto (\mathbb{N}, \leq),$$

- a **totally ordered set of labels** on the set of rewriting rules Σ_2

$$\psi : \quad \Sigma_2 \quad \longmapsto \quad \mathbb{N}$$

- For $wfw' \in \Sigma_{step}$, with $w, w' \in [n]^*$ and $f \in \Sigma_2$

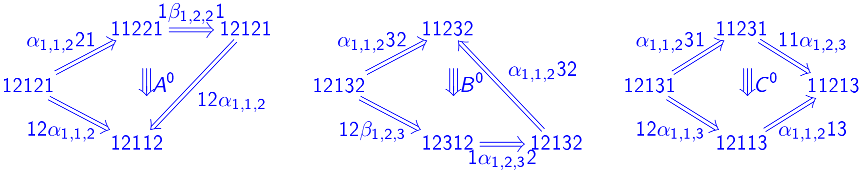
$$\psi(wfw') = (|w| - 1, \psi(f)),$$

  with $|1_{[n]^*}| = 0$ and $|w|$ indicates the length of w.

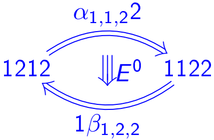
Application: plactic monoid P_n

Extension of $\Sigma(\mathrm{P}_n)$ by

- **generating $\mathcal{K}$-decreasing hw-confluence diagrams**



- a **generating hw-cycle**



Application: plactic monoid P_n

- The **extended presentation** of P_n

  $$\langle 1, \ldots, n \mid \alpha_{1,1,2}, k(\alpha_{1,1,2}) \mid A^0, k(A^0), B^0, k(B^0), C^0, k(C^0), E^0, k(E^0) \rangle.$$

  with k a **a sequence of Kashiwara's operators**, is coherent.

- By **homotopical reduction**:

  $$\langle 1, \ldots, n \mid \alpha_{1,1,2}, k(\alpha_{1,1,2}) \mid \cancel{A^0}, \cancel{k(A^0)}, \cancel{B^0}, \cancel{k(B^0)}, C^0, k(C^0), E^0, k(E^0) \rangle.$$

- A coherent presentation of P_n

  $$\langle 1, \ldots, n \mid \alpha_{1,1,2}, k(\alpha_{1,1,2}) \mid C^0, k(C^0), E^0, k(E^0) \rangle.$$

  with k a **a sequence of Kashiwara's operators**

V. Perspectives

Perspectives

- Applying our result of coherence by decreasingness to other algebraic structures with quasi-terminating presentations.

- apply decreasingness techniques to study Chinese monoids, [Endrullis-Klop, '19].