Operads with compatible CL-shellable partition poset admit a PBW basis

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Algebraic Rewriting Seminar
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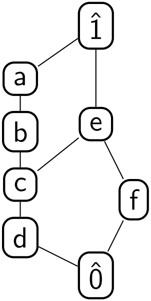
Outline

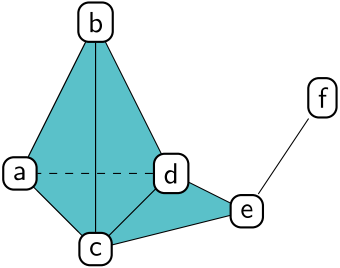
Posets and associated structures

Outline

Order complex

P poset $\quad\leftrightarrow\quad$ $\Delta(P)$ associated order complex





Simplicial complex (set of vertices V and faces $\Sigma \subseteq \mathcal{P}(V)$, stable by inclusion) defined as :

$$\Delta(P) = \{a_0 < \ldots < a_k | a_i \in P - \{\hat{0}, \hat{1}\}\}$$

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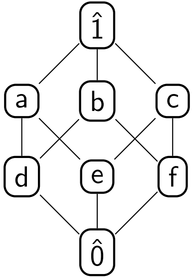

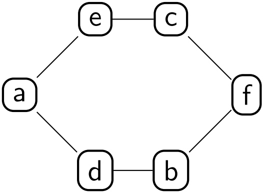

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Homology of a poset

P poset $\leftrightarrow \Delta(P)$ associated order complex

$$\Delta(P) = \{a_0 < \ldots < a_k | a_i \in P - \{\hat{0}, \hat{1}\}\}$$

$$C_k = Vect_{\mathbb{C}}(a_0 < \ldots < a_k | a_i \in P - \{\hat{0}, \hat{1}\})$$

$$\partial_k(a_0 \prec \ldots \prec a_k) = \sum_{i=0}^{k} (-1)^i (a_0 \prec \ldots \prec \hat{a}_i \prec \ldots \prec a_k)$$

$$C_{-1} = \mathbb{C}.e \xleftarrow{\partial_0} C_0 \xleftarrow{\partial_1} \ldots \xleftarrow{\partial_k} C_k \xleftarrow{\partial_{k+1}} C_{k+1} \xleftarrow{\partial_{k+2}} \ldots$$

$$\tilde{H}_j(P) = \tilde{H}_j(\Delta(P)) = \mathrm{Ker}\, \partial_j / \mathrm{Im}\, \partial_{j+1}$$

Cohen-Macaulay Poset : $\exists! j : \tilde{H}_j(P) \neq 0$

To keep in mind

Cohen-Macaulay posets are homotope to a bunch of spheres of same dimensions.

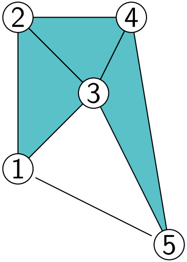
Shellable posets

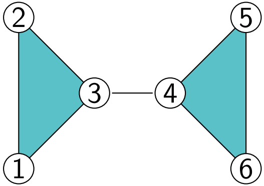
For F a face in $\Delta(P)$, let us set $\langle F \rangle = \{G : G \subseteq F\}$.

Definition

A poset is shellable if

- there exists an order on its facets $F_1, \ldots, F_t$ s.t.
- $\left(\cup_{i=1}^{k-1} \langle F_i \rangle\right) \cap \langle F_k \rangle$ est pur (facettes de même dim.)
- et de dimension $\dim F_k - 1$, $\forall k \in \{2, \ldots, t\}$.





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Proposition (Folklore, Björner 1980)

shellable $\implies$ *Cohen-Macaulay*

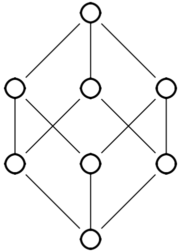
Combinatorial criterion : CL-shellability [Björner-Wachs, 1982]

CL-shellability = Chain Lexicographic-shellability

Definition

A maximal chain is a chain which is not strictly contained in another chain.
Given r a maximal chain of $[\hat{0}, x]$ and $[x, y]$ an interval, we can define the closed rooted interval

$$[x, y]_r := \{z \in r\} \cup \{z \in [x, y]\}.$$



CL-shellability [Björner-Wachs, 1982]

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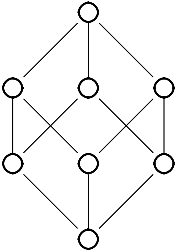

CL-shellability [Björner-Wachs, 1982]

Definition

A Chain-Lexicographic labelling of Π = chain-edge labelling $\lambda : ME(\Pi) \to \Lambda$ s.t.

- in each $[x, y]_r$, $\exists! c$ maximal chain whose associated labels forms a strictly increasing chain in Λ.
- c precedes lexicographically all the tuples associated with other maximal chains of $[x, y]_r$

A poset that admits a CL-labelling is said to be CL-shellable.

Proposition (Björner-Wachs 1982)

CL-shellable $\implies$ Cohen-Macaulay

CL-shellability [Björner-Wachs, 1982]

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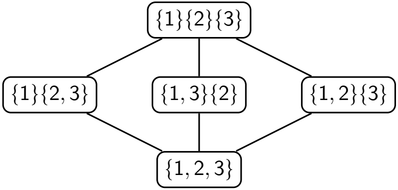

CL-shellability [Björner-Wachs, 1982]

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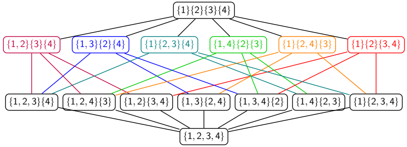

CQVDR

CL-shellable $\Longrightarrow$ shellable $\Longrightarrow$ Cohen-Macaulay

Shuffle operads and PBW basis

Outline

Shuffle operads [Dotsenko-Khoroshkin, 2010]

Definition

An ordered species is a functor $\mathcal{O} : Ord \to Set$.
On $\mathcal{O}$ and $\mathcal{Q}$ two ordered species, define

$$\mathcal{O} \circ_{sh} \mathcal{Q}(E) = \bigcup_k \mathcal{O}([k]) \times \left(\bigcup_{w \in \mathrm{Sh}(E,[k])} \bigtimes_{i=1}^{k} \mathcal{Q}(w^{-1}(i)) \right),$$

where $\mathrm{Sh}(E,[k])$ is the set of surjections $w : E \to [k]$ satisfying the *shuffle condition*

$$\min w^{-1}(1) < \min w^{-1}(2) < \ldots < \min w^{-1}(k).$$

$$I : E \mapsto E \text{ if } |E| = 1, \varnothing \text{ otherwise.}$$

Shuffle operads [Dotsenko-Khoroshkin, 2010]

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Definition (Dotsenko - Khoroshkin 10)

A shuffle operad is an ordered species $\mathcal{O}$ endowed with an associative product $\mu : \mathcal{O} \circ_{sh} \mathcal{O} \to \mathcal{O}$ and a unit $\nu : I \to \mathcal{O}$.

Free shuffle operad

Let $E = (E_n)_{n \geqslant 1}$ be a graded set, $1 \in E_1$,

Definition

$\mathcal{T}(E)$ = free shuffle operad

$\bigotimes_{d \geqslant 1} \mathcal{T}(E)(S)^{(d)}$ is spanned by PBT s.t.

- d leaves labelled by S
- inner nodes of arity k labelled by E_k
- smallest leaf always in the left subtree

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- smallest leaf always in the left subtree

Here,

- Connected operads: $E(n) = 0$ for $n = 0$ and $n = 1$
- Associative algebras: $E(n) = 0$ for $n \neq 1$

Normalised reduced bar construction [Fresse 04]

$\mathcal{N}_l(\mathcal{P}) = \mathbb{S}$-module represented by non-degenerate l-levelled trees.

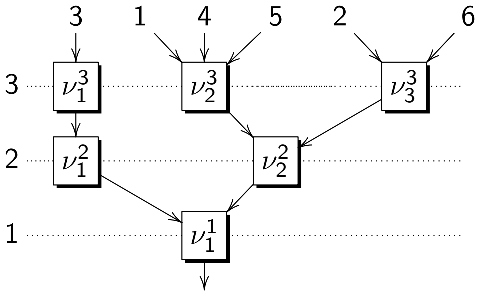


Figure: Non-degenerate 3-levelled tree

$+$ differential given by contracting inner edges.

Koszul

Definition

An operad is Koszul if the homology of its bar construction is concentrated in maximal degree.

PBW property (for algebraic operads)

$\mathcal{B}^E$ k-linear basis of E (in Vect)
$\mathcal{B}^{\mathcal{T}(E)}$ associated (monomial) basis of $\mathcal{T}(E)$.
Assume that $\mathcal{B}^E$ is partially ordered, in a way compatible with the arity:

$$\mu < \nu \text{ if } \mu \in \mathcal{B}^E(k) \text{ and } \nu \in \mathcal{B}^E(l) \text{ with } k < l.$$

Extension to $\mathcal{B}^{\mathcal{T}(E)}$ s.t.: for $\alpha, \alpha' \in \mathcal{T}(E)(S_1)$ and $\beta, \beta' \in \mathcal{T}(E)(S_2)$,

$$\left\{ \begin{array}{l} \alpha \leqslant \alpha' \\ \beta \leqslant \beta' \end{array} \right. \Rightarrow \forall w \text{ pointed shuffle, } \alpha \circ_w \beta \leqslant \alpha' \circ_w \beta'.$$

(compatibility with the composition)

PBW property (for algebraic operads)

Definition

A Poincaré-Birkhoff-Witt (PBW) basis for $\mathcal{P} = \mathcal{T}(E)/(R)$ is a subset $\mathcal{B}^{\mathcal{P}}$ of $\mathcal{B}^{\mathcal{T}(E)}$ such that

- $1 \in \mathcal{B}^{\mathcal{P}}$,
- $\mathcal{B}^{E} \subset \mathcal{B}^{\mathcal{P}}$,
- $\mathcal{B}^{\mathcal{P}}$ represents a basis of the $\mathbb{K}$-module $\mathcal{P}$,

and satisfying the following conditions:

1. for $\alpha, \beta \in \mathcal{B}^{\mathcal{P}}$ either $\alpha \circ_{i,w} \beta \in \mathcal{B}^{\mathcal{P}}$, or $\alpha \circ_{i,w} \beta = \sum_{\gamma} c_{\gamma}\gamma$, where the $\gamma \in \mathcal{B}^{\mathcal{P}}$ and $\gamma < \alpha \circ_{i,w} \beta$ in $\mathcal{T}(E)$;
2. $\alpha \in \mathcal{B}^{\mathcal{P}}$ of shape τ if and only $\forall e \in \tau$, $\alpha_{|\tau_e} \in \mathcal{B}^{\mathcal{P}}$.

PBW property

Theorem (consequence of Hoffbeck 10)

An operad equipped with a partially ordered PBW basis is Koszul.

CQVDR

PBW $\Longrightarrow$ Koszul

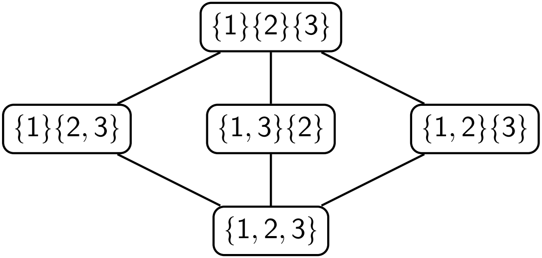
Operadic Partition Posets

Outline

(Set) Partition poset Π_3

$$
\begin{tikzcd}
 & \{1\}\{2\}\{3\} \arrow[dl, no head] \arrow[d, no head] \arrow[dr, no head] & \\
\{1\}\{2,3\} \arrow[dr, no head] & \{1,3\}\{2\} \arrow[d, no head] & \{1,2\}\{3\} \arrow[dl, no head] \\
 & \{1,2,3\} &
\end{tikzcd}
$$

(Set) Partition poset Π_3

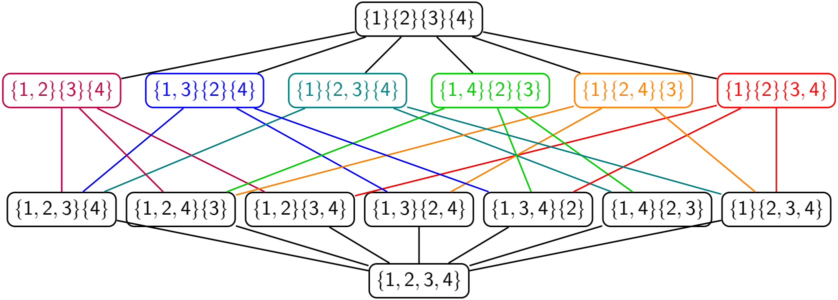


$$\pi = \{\pi_1, \ldots, \pi_k\} \leqslant \{\mu_1, \ldots, \mu_p\} = \mu$$

$$\Leftrightarrow$$

$$\pi_i = \cup_{j=1}^{n_i} \mu_{i_j}$$

Partition poset Π_4



Homology of a poset

Theorem (Stanley 82, Hanlon 81,Joyal 85)

$$H_{n-1}(\Pi_n) = \mathrm{Lie}(n)^* \otimes \mathrm{sgn}_n$$

[Fresse 04] $\rightsquigarrow$ Link with the Koszul theory for operad

Generalized partition poset [Vallette 07]

Definition (Joyal 80)

A set species is a functor $\mathcal{O} : Set \to Set$.
On $\mathcal{O}$ and $\mathcal{Q}$ two set species, define

$$\mathcal{O} \circ \mathcal{Q}(E) = \bigcup_{\pi \in \Pi(E)} \mathcal{O}(\pi) \times \bigtimes_{p \in \pi} \mathcal{Q}(p)$$

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Definition

A set operad is a set species $\mathcal{O}$ endowed with an associative product $\mu : \mathcal{O} \circ \mathcal{O} \to \mathcal{O}$ and a unit $\nu : I \to \mathcal{O}$.

It is basic-set if the map $\mu_{(\nu_1, \ldots, \nu_t)} : \nu \to \mu(\nu(\nu_1, \ldots, \nu_t))$ is injective.

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Definition

A partition decorated by an operad $\mathcal{O}$ is a partition $\pi = \{\pi_1,\ldots,\pi_n\}$ with a choice for each part π_i of a $\pi_i^{\mathcal{O}} \in \mathcal{O}(\pi_i)$

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Example

- $\mathcal{O} = \mathrm{Comm} \rightsquigarrow$
- $\mathcal{O} = \mathrm{Ass} \rightsquigarrow$
- $\mathcal{O} = \mathrm{Lie} \rightsquigarrow$

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- $\mathcal{O} = \mathrm{Ass} \rightsquigarrow$ poset of partition with ordered parts
- $\mathcal{O} = \mathrm{Lie} \rightsquigarrow$ it is not a set operad !

Generalized partition poset [Vallette 07]

$\Pi_{\mathcal{O}}$

$$\pi^{\mathcal{O}} = \{\pi_1^{\mathcal{O}}, \ldots, \pi_k^{\mathcal{O}}\} \leqslant \{\mu_1^{\mathcal{O}}, \ldots, \mu_p^{\mathcal{O}}\} = \mu^{\mathcal{O}}$$

$$\Leftrightarrow$$

$$\pi_i^{\mathcal{O}} = \mu(\nu(\mu_{i_1}, \ldots, \mu_{i_{n_i}}))$$

Example for $\mathcal{O} = Assoc$

$$(1547)(62)(38) \leqslant (62154738), (38621547), \ldots$$

$$\nleqslant (15642738)$$

Generalized partition poset [Vallette 07]

Theorem (Vallette 07)

Operad $\mathcal{O}$ is Koszul if and only if each subposet $[\alpha, \hat{1}]$ of each $\Pi_{\mathcal{O}}(n)$ is Cohen-Macaulay for $\alpha \in \mathcal{O}(\{1, \ldots, n\})$.

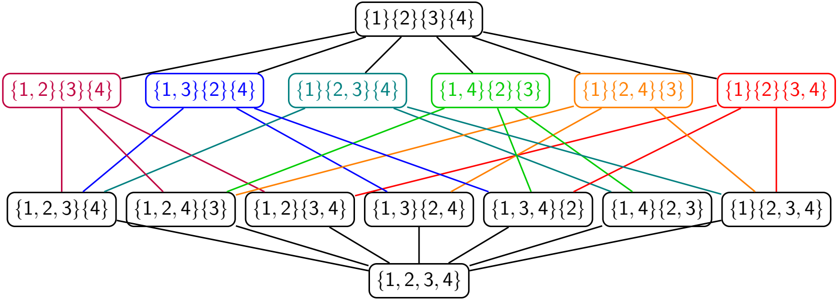
Question ?

$$\begin{array}{ccc} \text{CL-shellable} & & \\ \Downarrow & & \\ \text{shellable} & & \text{PBW} \\ \Downarrow & & \Downarrow \\ \text{Cohen-Macaulay} & \Longleftrightarrow & \text{Koszul} \end{array}$$

Question

To which poset property correspond PBW property ?

Partition poset Π_4



Our answer to the question

Outline

Main theorem

Theorem (B.M.-D.O.-H., 21)

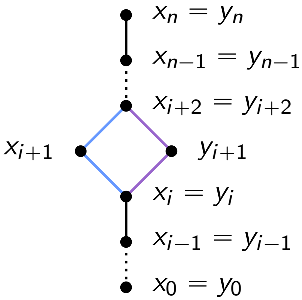
$\widetilde{\mathcal{P}}$ = quadratic basic-set operad
$\Pi_{\widetilde{\mathcal{P}}}$ = operadic partition posets
$\Pi_{\widetilde{\mathcal{P}}}^{(d)} = \{\lambda \in \mathcal{P} : \exists \nu \in \tilde{\mathcal{P}}^{(d)}$ such that $\lambda \leqslant \bar{\nu}\}$ admit CL-labellings compatible with isomorphisms of subposets

$$\Downarrow$$

Then, the algebraic operad $\mathcal{P} = \mathcal{T}(E)/(R)$ associated to $\widetilde{\mathcal{P}}$ admits a PBW basis with a partial order.

Let us detail our order !

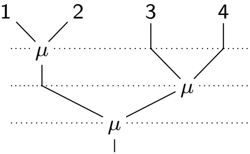
Adjacent chains

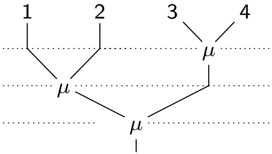


Two maximal chains are adjacent if they only differs by two edges.

Exchange relation

Exchange relation between $\tilde{a} = \nu^1 \circ_{i_1} \nu^2 \circ_{i_2} \cdots \circ_{i_{l-1}} \nu^l$ and $\tilde{b} = \mu^1 \circ_{j_1} \mu^2 \circ_{j_2} \cdots \circ_{j_{m-1}} \mu^m$ in $\mathcal{N}(E)$ if $\pi(\tilde{a}) = \pi(\tilde{b})$ and if there exists $k \in [\![1,\ l-1]\!]$ such that $\nu^s = \mu^s$ and $i_s = j_s$ for all $s \in [\![1,\ l]\!] \setminus \{k,\ k+1\}$.

$\tilde{a} =$ 

and $\tilde{b} =$ 

Definition of the partial order

Definition

For $a, b \in \mathcal{T}(E)$, $a \neq b$,
$a \lessdot b$ if there exist two adjacent chains $\tilde{a} = r \cdot (g \prec x \prec h) \cdot s$ and $\tilde{b} = r \cdot (g \prec y \prec h) \cdot s$ such that:

- $a = \pi(\tilde{a})$ and $b = \pi(\tilde{b})$,
- the CL-labelling given by $\lambda_r^{g,h}$ of the chain $(g \prec x \prec h)$ is the unique increasing chain (minimal in the lexicographic order) in this interval.

Definition of the partial order

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Lemma

$\lessdot$ is anti-symmetric as soon as $\Pi_{\tilde{\mathcal{P}}}$ admit CL-labellings compatible with the isomorphisms of subposets (iso-CL-labellings).

Lemma

$\Pi_{\tilde{\mathcal{P}}}$ admit iso-CL-labellings. The reflexive and transitive closure of the relation $\lessdot$, denoted by $\leqslant$, satisfies that

$$a < b \implies \min\left(\{\tilde{a}|\pi(\tilde{a}) = a\}\right) <_{lex} \min\left(\left\{\tilde{b}|\pi(\tilde{b}) = b\right\}\right),$$

where

- *$<_{lex}$ = lexicographic order on chains,*
- *min is the minimum for $<_{lex}$,*
- *only maximal chains.*

Hence, $\leqslant$ is a well-defined partial order.

Minimal elements : a s.t. one of its representatives is the minimal increasing chain in the interval.

Main theorem

Theorem

$\widetilde{\mathcal{P}}$ *= quadratic basic-set operad*
$\Pi_{\widetilde{\mathcal{P}}}$ *= operadic partition posets*
$\left\{\Pi_{\widetilde{\mathcal{P}}}^{(d)}\right\}_d$ *admit CL-labellings compatible with isomorphisms of subposets*

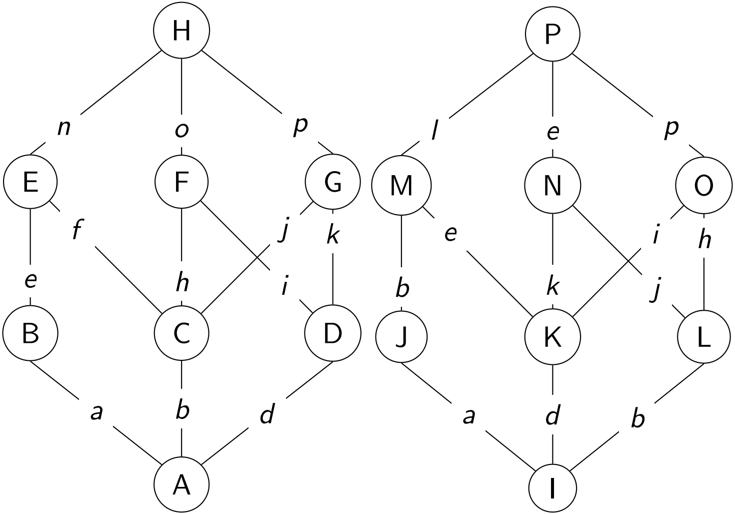
$$\Downarrow$$

Then, the algebraic operad $\mathcal{P} = \mathcal{T}(E)/(R)$ *associated to* $\widetilde{\mathcal{P}}$ *admits a PBW basis with a partial order.*

Corollaries

Recover usual PBW basis for Comm and Perm.

Counter-example to the converse



Consider the algebra on 13 generators: a, b, d, e, f, h, i, j, k, l, n, o, p, with relations given as follows.

$$\begin{aligned} ba &= ed & ea &= fb \\ hb &= id & jb &= kd \\ oi &= pk & ej &= ph \\ le &= ek = pi & nf &= oh = pj \end{aligned}$$

Thank you very much for your attention !