Homology of strict ω-categories

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Nerve of ω-categories

Definition

The nerve of an ω-category C is the simplicial set

$$\begin{aligned} N_\omega(C) : \Delta^{\mathrm{op}} &\to \mathsf{Set} \\ [n] &\mapsto \mathrm{Hom}_{\omega\mathsf{Cat}}(\mathcal{O}_n, C). \end{aligned}$$

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Example

When C is a (1-)category, $N_\omega(C)$ is nothing but the usual nerve of C.

Singular homology

Recall that to each simplicial set X, we can associate a chain complex

$$K(X) = \mathbb{Z}X_0 \longleftarrow \mathbb{Z}X_1 \longleftarrow \mathbb{Z}X_2 \longleftarrow \cdots$$

which allows to define the homology groups of X as the homology groups of the chain complex $K(X)$.

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Definition

Let C be an ω-category. The singular homology groups of C are the homology groups of its nerve $N_\omega(C)$.

Polygraphs

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An ω-category is free on a polygraph if it can be obtained recursively from the empty category by freely attaching cells.

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If C is a ω-category free on a polygraph, then there is a *unique* set of generating cells possible.

Terminological convention:

$$\text{polygraph} = \omega\text{-category free on a polygraph.}$$

Polygraphic homology

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Polygraphs $\cong$ CW-complexes
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(Remark: Later we will see how to define polygraphic homology for all ω-categories, not just polygraphs.)

A natural question:

Let C be a ω-category (free on a polygraph). Do we have
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Polygraphic homology vs singular homology

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Short answer: Not always. It depends on C.

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Short answer: Not always. It depends on C.
(Hence, polygraphic homology doesn't work as well as cellular homology of CW-complexes.)

Ara and Maltsiniotis' counter-example

Let B be the 2-polygraph

- one object: $\bullet$,
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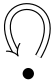
$B =$ 

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We have

$$H_k^{\mathrm{pol}}(B) \simeq \begin{cases} \mathbb{Z} & \text{if } k = 0, 2 \\ 0 & \text{otherwise.} \end{cases}$$

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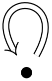
$B =$ 

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Conclusion :

$$H_{2p}^{\mathrm{pol}}(B) \not\simeq H_{2p}^{\mathrm{Sing}}(B) \text{ for } p \geq 2.$$

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The fundamental question

For which ω-categories C do we have $H_{\bullet}^{\mathrm{pol}}(C) \simeq H_{\bullet}^{\mathrm{Sing}}(C)$?

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The fundamental question

For which ω-categories C do we have $H_{\bullet}^{\mathrm{pol}}(C) \simeq H_{\bullet}^{\mathrm{Sing}}(C)$?

This is what I tried to answer in my PhD.

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Homotopy theory: quick recollection

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When so, we can construct the homotopy category

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Intuition: $\mathcal{H}\mathrm{o}(\mathcal{C})$ is the category of objects of $\mathcal{C}$ "up to morphisms of $\mathcal{W}$".
Remark: Usually, we'll have more than just $\mathcal{C}$ and $\mathcal{W}$ (for example a model structure); but not always.

Thomason homotopy theory

Definition

A morphism $f\colon C \to D$ of $\omega\mathrm{Cat}$ is a Thomason equivalence if $N_\omega(f)\colon N_\omega(C) \to N_\omega(D)$ is a weak equivalence of simplicial sets.

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Let us write

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By definition, the nerve functor induces

$$\overline{N_\omega} : \mathcal{H}\mathrm{o}(\omega\mathsf{Cat}^{\mathrm{Th}}) \to \mathcal{H}\mathrm{o}(\widehat{\Delta}).$$

Theorem (Gagna, 2018)

$\overline{N_\omega} : \mathcal{H}\mathrm{o}(\omega\mathsf{Cat}^{\mathrm{Th}}) \to \mathcal{H}\mathrm{o}(\widehat{\Delta})$ is an equivalence of categories (or better an equivalence of derivators, or of weak $(\infty, 1)$-categories).

ω-categories as spaces

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In other words:

Homotopy theory of ω-categories induced by Thomason equivalences

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Hence,

Singular homology of ω-categories
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(Co-inductive definition.)

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Example: When C and D are (1-)categories, we recover the usual notion of equivalence of categories.

Globes and spheres

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etc.

- let $i_n : \mathbb{S}_{n-1} \to \mathbb{D}_n$ be the "boundary" inclusion.

The folk model structure

Theorem (Lafont,Métayer,Worytkiewicz - 2010)

There exists a model structure on ωCat such that:

- the weak equivalences are the equivalences of ω-categories,
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Theorem (Métayer - 2008)

The cofibrant objects of the folk model structure are exactly the polygraphs.

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Best case scenario: F preserves weak equivalences, thus it induces a canonical functor

$$\overline{F} : \mathcal{H}\mathrm{o}(\mathcal{C}) \to \mathcal{H}\mathrm{o}(\mathcal{C}'),$$

Derived functors: quick recollection

Let $(\mathcal{C}, \mathcal{W})$ and $(\mathcal{C}', \mathcal{W}')$ be two categories equipped with weak equivalences and $F : \mathcal{C} \to \mathcal{C}'$ a functor.
Best case scenario: F preserves weak equivalences, thus it induces a canonical functor

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<u>Usual scenario:</u> F does not preserves weak equivalences, but we can still construct a functor $\mathcal{H}\mathrm{o}(\mathcal{C}) \to \mathcal{H}\mathrm{o}(\mathcal{C}')$.

Definition

Let $(\mathcal{C}, \mathcal{W})$ and $(\mathcal{C}', \mathcal{W}')$ be two categories equipped with weak equivalences and $F : \mathcal{C} \to \mathcal{C}'$ a functor. The functor F is left derivable if there exists a functor $\mathbb{L}F : \mathcal{H}\mathrm{o}(\mathcal{C}) \to \mathcal{H}\mathrm{o}(\mathcal{C}')$ and a natural transformation α

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Model category theory provides tools to compute derived functors.

Abelianization of ω-categories

Recall that there is a functor

$$\lambda : \omega\mathsf{Cat} \to \mathsf{Ch}_{\geq 0},$$

referred to as the abelianization functor. For an ω-category C, the chain complex $\lambda(C)$ is defined as:

- $\lambda(C)_n = \mathbb{Z}C_n/{\sim}$, where $\sim$ is generated by

$$x \underset{k}{*} y \sim x + y,$$

  whenever x and y are k-composable,
- $\partial : \lambda(C)_n \to \lambda(C)_{n-1}$ is the only linear map such that

$$\partial(x) = \mathrm{t}(x) - \mathrm{s}(x)$$

  for every n-cell x.

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Important: When C is a polygraph, $\lambda(C)$ is exactly the chain complex used to compute the polygraphic homology of C.

$\mathcal{H}\mathrm{o}(\mathsf{Ch}_{\geq 0}) :=$ localization of $\mathsf{Ch}_{\geq 0}$ with respect to quasi-isomorphisms.

Polygraphic homology as derived functor

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Proposition (folklore ?)

The functor $\lambda : \omega\mathsf{Cat} \to \mathsf{Ch}_{\geq 0}$ is left Quillen w.r.t the folk model structure on $\omega\mathsf{Cat}$ and the projective model structure on $\mathsf{Ch}_{\geq 0}$.

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In particular it is left derivable

$$\mathbb{L}\lambda^{\mathrm{folk}} \colon \mathcal{H}\mathrm{o}(\omega\mathsf{Cat}^{\mathrm{folk}}) \to \mathcal{H}\mathrm{o}(\mathsf{Ch}_{\geq 0}).$$

Moreover, for every ω-category C (free on a polygraph) and every $k \geq 0$, we have

$$H_k^{\mathrm{pol}}(C) \simeq H_k(\mathbb{L}\lambda^{\mathrm{folk}}(C)).$$

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From now on, we define the polygraphic homology functor $\mathbb{H}^{\mathrm{pol}}$ as:

$$\mathbb{H}^{\mathrm{pol}} := \mathbb{L}\lambda^{\mathrm{folk}} \colon \mathcal{H}\mathrm{o}(\omega\mathsf{Cat}^{\mathrm{folk}}) \to \mathcal{H}\mathrm{o}(\mathsf{Ch}_{\geq 0}).$$

Polygraphic homology for all

Note: We have extended the definition of polygraphic homology from polygraphs to *all* ω-categories. When an ω-category C is not a polygraph, it suffices to a take cofibrant replacement (=polygraphic resolution) of C.

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Let M be a monoid (considered as an ω-category). We have

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Historically, this was the motivation for polygraphic homology.

Singular homology as a derived functor

Theorem (G. - 2020)

The functor $\lambda\colon \omega\mathsf{Cat} \to \mathsf{Ch}_{\geq 0}$ is left derivable w.r.t the *Thomason equivalences* on $\omega\mathsf{Cat}$

$$\mathbb{L}\lambda^{\mathrm{Th}}\colon \mathcal{H}\mathrm{o}(\omega\mathsf{Cat}^{\mathrm{Th}}) \to \mathcal{H}\mathrm{o}(\mathsf{Ch}_{\geq 0}),$$

and for every ω-category C and every $k \geq 0$, we have

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Conclusion: The polygraphic homology and the singular homology are obtained as left derived functors of the same functor, but not w.r.t to the same weak equivalences !

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Equivalence of ω-categories vs Thomason equivalences

Important Lemma

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<u>Remark</u>: The converse of the above lemma is false. For example

$$\mathbb{D}_1 \to \mathbb{D}_0$$

is a Thomason equivalence but not an equivalence of ω-categories.

Canonical comparison map

Proposition (abstract non-sense)

There is a canonical natural transformation

$$\begin{tikzcd}
\mathcal{H}\mathrm{o}(\omega\mathsf{Cat}^{\mathrm{folk}}) \arrow[d, "\mathcal{J}"'] \arrow[dr, "\mathbb{H}^{\mathrm{pol}}"] & \\
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In other words, for every ω-category C we have a map

$$\pi_C : \mathbb{H}^{\mathrm{Sing}}(C) \to \mathbb{H}^{\mathrm{pol}}(C),$$

which is natural in C. We refer to it as the canonical comparison map.

Homologically coherent ω-categories

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Definition

An ω-category C is homologically coherent if the map

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Goal: Understand which ω-categories are homologically coherent.

Polygraphic homology is not homotopical

Another formal consequence of the formalism of left derived functors:

Proposition (abstract non-sense)

There exists at least one Thomason equivalence $u : C \to D$ such that the induced morphism

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New slogan

The polygraphic homology is a way of computing the singular homology of homologically coherent ω-categories.

Side note: equivalence of homologies in low dimension

Proposition

Let C be *any* ω-category. The canonical comparison map induces an isomorphism

$$H_k^{\mathrm{Sing}}(C) \to H_k^{\mathrm{pol}}(C)$$

for $k = 0, 1$.

Side note: equivalence of homologies in low dimension

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Open question:

Do we have

$$H_k^{\mathrm{pol}}(C) \simeq H_k^{\mathrm{Sing}}(C)$$

for $k = 2, 3$, for any ω-category C ?

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We equip the category $\mathcal{C}^I$ of functors from I to $\mathcal{C}$ with the class of pointwise weak equivalences.

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We say that $\mathcal{C}$ has homotopy colimits (with respect to $\mathcal{W}$) of shape I if the colimit functor

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Similarly to usual category theory, in homotopy theory there is a notion of functors that *preserves homotopy colimits*.

Back to the triangle:

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Idea: exploit that sometimes it *is* an isomorphism.

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Then C is homologically coherent.

Easy application: homology of globes and spheres

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The image by N_ω of $(*)$ in $\widehat{\Delta}$ is a *cocartesian* square of monos, hence homotopy cocartesian.

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By an immediate induction, $\mathbb{S}_n$ is homologically coherent (and has the homotopy type of an n-sphere).

The case of 1-categories

Theorem (G. - 2019)

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Remark 1: The homology (polygraphic or singular) of a category need not be trivial above dimension 1.
Remark 2: Extension of Lafont and Métayer's result on the homology of monoids, but more precise and completely new proof.

The case of 1-categories

Sketch of proof: Let A be a small category. Recall that

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Too long to explain but uses crucially the notion discrete Conduché ω-functors (invented for this purpose).

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- Archetypal situation to understand: given a cocartesian square

$$\begin{tikzcd} \mathbb{S}_1 \arrow[r] \arrow[d, "i_1"'] & P \arrow[d] \\ \mathbb{D}_2 \arrow[r] & P' \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

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with P and P' 2-polygraphs, when is it homotopy cocartesian w.r.t the Thomason equivalences ?

I do not have a general answer to this question...

However, using tools that I don't have time to explain, I know how to answer this question in many concrete situations.

Zoology of 2-categories: basic examples

For $n, m \geq 0$, let $A_{(m,n)}$ be the 2-polygraph, with one generating 2-cell whose source is a chain of length m and target a chain of length n:

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Examples:

- $A_{(1,1)}$ is $\mathbb{D}_2$.

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Examples:

- $A_{(1,1)}$ is $\mathbb{D}_2$.
- $A_{(0,0)}$ is the 2-polygraph B from Ara and Maltsiniotis' counter-example.

Proposition

If $n + m > 0$, the 2-category $A_{(m,n)}$ has the homotopy type of a point and is homologically coherent.
Else, $A_{(0,0)}$ has the homotopy type of a $K(\mathbb{Z}, 2)$.

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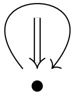
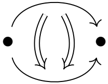
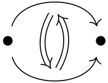
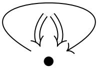
Remark: not that obvious for $m + n = 1$.
Example:

$A_{(1,0)} =$ 

has many non-trivial 2-cells.

Zoology of 2-categories: variation of spheres

2-category	homologically coherent?	homotopy type
	yes	$\mathbb{S}_2$
	yes	$\mathbb{S}_2$
	yes	$\mathbb{S}_2$
	yes	$\mathbb{S}_2$
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Zoology of 2-categories: Bouquets of spheres

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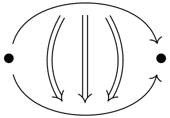
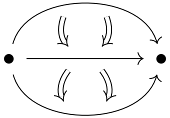
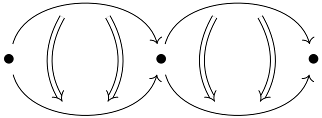
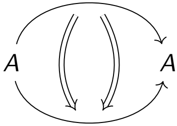
	yes	$\mathbb{S}_2 \vee \mathbb{S}_2$
	yes	$\mathbb{S}_2 \vee \mathbb{S}_2$
	yes	$\mathbb{S}_2 \vee \mathbb{S}_2$
	yes	$\mathbb{S}_2 \vee \mathbb{S}_1$

Let C be the free 2-polygraph generated by

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & A \arrow[d, "g"] \\ A \arrow[r, "f"'] \arrow[ur, Rightarrow] & A. \end{tikzcd}$$

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This 2-polygraph has the homotopy type of the torus and is homologically coherent.

Bubbles

Definition

A bubble in a 2-category is a non unit 2-cell α whose source and target are units on a 0-cell.

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In pictures:

$$\begin{tikzcd} A \arrow[r, bend left=60, "1_A"] \arrow[r, bend right=60, "1_A"'] \arrow[r, phantom, "\Downarrow\alpha"] & A \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} A \arrow[Rightarrow, loop above, "\alpha"] \end{tikzcd}.$$

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Definition

A 2-category is bubble-free if it has no bubbles.

The bubble-free conjecture

The archetypal example of *non* bubble-free 2-polygraph is the 2-category B from Ara and Maltsiniotis' counter-example.

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Conjecture

Let C be a 2-polygraph. It is homologically coherent if and only if it is bubble-free.

Merci pour votre attention !